1 **(10 marks)**

1.(a) **(5 marks)** Find the domain of the function

$$f(x) = \frac{3x + |x|}{\sqrt[4]{x^2 - 5x}}.$$

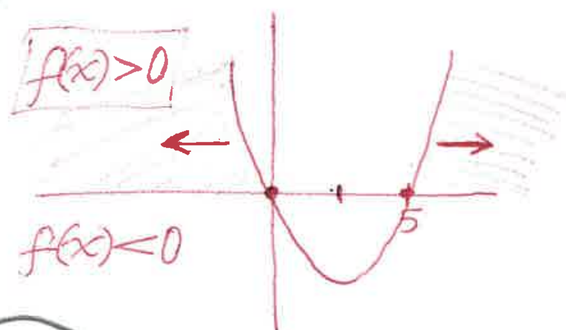
$x^2 - 5x > 0$

$x(x-5) > 0$

1 mark

	0		5	
x	$-$	$+$		$+$
$x-5$	$-$	$-$		$+$
x^2-5x	$+$	$-$		$+$

OR



$D_f = (-\infty, 0) \cup (5, +\infty)$

4 marks

1.(b) **(5 marks)** For a given function f show that the function h defined by

$$h(x) = f(x) - f(-x)$$

is an odd function.

$$\left.\begin{aligned} h(-x) &= f(-x) - f(-(-x)) \\ &= f(-x) - f(x) \end{aligned}\right\} \text{2 marks}$$

$$= -(-f(-x) + f(x))$$

$$= -(f(x) - f(-x)) \qquad \text{2 marks}$$

$$= -h(x) \qquad \text{1 mark}$$

- No more than 1 mark if student treated f as an even or odd function (failing to consider that f could be neither).
- Adding $h(x)$ to both sides to get $h(-x)+h(x)=0$ is also a valid approach.

2 **(25 marks)**

2.(a) **(7 marks)** Let f be a function with domain D, where D is an open interval. Let

$$\lim_{x \to a} f(x) = L.$$

Give the ϵ, δ statement of the limit of f at a. Then use this definition to prove that

$$\lim_{x \to 1}(2x+1) = 3.$$

Def: For every $\varepsilon > 0$ there is $\delta > 0$ such that

$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$ 3 marks

Let $\varepsilon > 0$.

$|2x+1-3| = |2x-2| = 2|x-1| < \varepsilon$

$|x-1| < \frac{\varepsilon}{2}$

choose $\delta = \frac{\varepsilon}{2}$

$|x-1| < \delta = \frac{\varepsilon}{2} \implies 2|x-1| < \varepsilon \implies |2x-2| < \varepsilon$

$\implies |2x+1-3| < \varepsilon$ 4 marks

2.(b) **(18 marks)** State whether the limit exists; evaluate the limit if it does exist.

i. $\lim_{x\to 3+} \frac{\sqrt{x-3}}{|x-3|}$

ii. $\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

iii. $\lim_{x\to\infty} \frac{3e^x}{e^x-6}$

iv. $\lim_{x\to\infty} (\sqrt{x^2+1}-x)$

v. $\lim_{x\to\infty} x^{\frac{1}{x}}$

vi. $\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$, Here a and b are fixed numbers and $b \neq 0$

i

$$\lim_{x\to 3^+} \frac{\sqrt{x-3}}{|x-3|} = \lim_{x\to 3^+} \frac{\sqrt{x-3}}{x-3}$$

3 marks

$$= \lim_{x\to 3^+} \frac{1}{\sqrt{x-3}} = +\infty$$

ii

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} = \lim_{t\to 0} \frac{(\sqrt{1+t}-\sqrt{1-t})(\sqrt{1+t}+\sqrt{1-t})}{t(\sqrt{1+t}+\sqrt{1-t})}$$

$$= \lim_{t\to 0} \frac{1+t-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})} = \lim_{t\to 0} \frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})} = 1$$

3 marks

iii

$$\lim_{x\to\infty} \frac{3e^x}{e^x-6} = \lim_{x\to\infty} \frac{3\cancel{e^x}}{\cancel{e^x}(1-\frac{6}{e^x})} = \frac{3}{1-0} = 3$$

Sol 2 :

$$\lim_{x\to\infty} \frac{3e^x}{e^x-6} \overset{L'H}{=} \lim_{x\to\infty} \frac{3e^x}{e^x} = 3$$

3 marks

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(iv) $$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) = \lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)}$$

$$= \lim_{x\to\infty}\frac{x^2+1-x^2}{\left(\sqrt{x^2+1}+x\right)}$$

$$= 0$$

3 marks

(v) $$\lim_{x\to\infty} x^{\frac{1}{x}}$$

$$x^{\frac{1}{x}} = y$$

$$\frac{1}{x}\ln x = \ln y$$

1 mark

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\ln y$$

$$\lim_{x\to\infty}\frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\ln y$$

1 mark

$$0 = \lim_{x\to\infty}\ln y \quad \Rightarrow \lim_{x\to\infty} y = 1$$

1 mark

$$\lim_{x\to\infty} x^{\frac{1}{x}} = 1$$

(vi) $$\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)} \overset{L'H}{=} \lim_{x\to 0}\frac{a\cos(ax)}{b\cos(bx)} = \frac{a}{b}$$

1 mark

2 marks

3 **(20 marks)**

3. (a) **(7 marks)** Let A, B be constants. For what values of A and B is the function f continuous on $(-\infty, \infty)$? Justify your answer.

$$f(x) = \begin{cases} Ax + B, & \text{if } x < -1 \\ 2x, & \text{if } -1 \le x \le 2 \\ 2Bx - A, & \text{if } x > 2 \end{cases}$$

f is continuous for all values $x \in (-\infty, -1) \cup (-1, +2) \cup (2, +\infty)$

(all the formulas that define functions are linear functions) 1 mark

At $x = -1$:

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x) = -2$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to (-1)^-} (Ax + B) = -A + B$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to (-1)^-} f(x)$$

2 marks

$$\boxed{-2 = -A + B}$$

At $x = 2$:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2Bx - A) = 4B - A$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x) = 4$$

2 marks

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \Rightarrow \boxed{4B - A = 4}$$

2 marks

$$\begin{cases} B - A = -2 \\ 4B - A = 4 \end{cases} \Rightarrow B - \cancel{A} - 4B + \cancel{A} = -2 - 4 \Rightarrow -3B = -6 \Rightarrow \boxed{B = 2}$$

$$2 - A = -2 \Rightarrow \boxed{A = 4}$$

3. (b) **(8 marks)** Suppose that f is a function such that $|f(x)| \leq |x|$ for all x. Show that f is continuous at zero.

$|f(x)| \leq |x|$ for all x. So $|f(0)| \leq |0| = 0$

Therefore $f(0) = 0$ 1 mark

By ε–δ def of limit for $\varepsilon > 0$ we need to find $\delta > 0$ such that

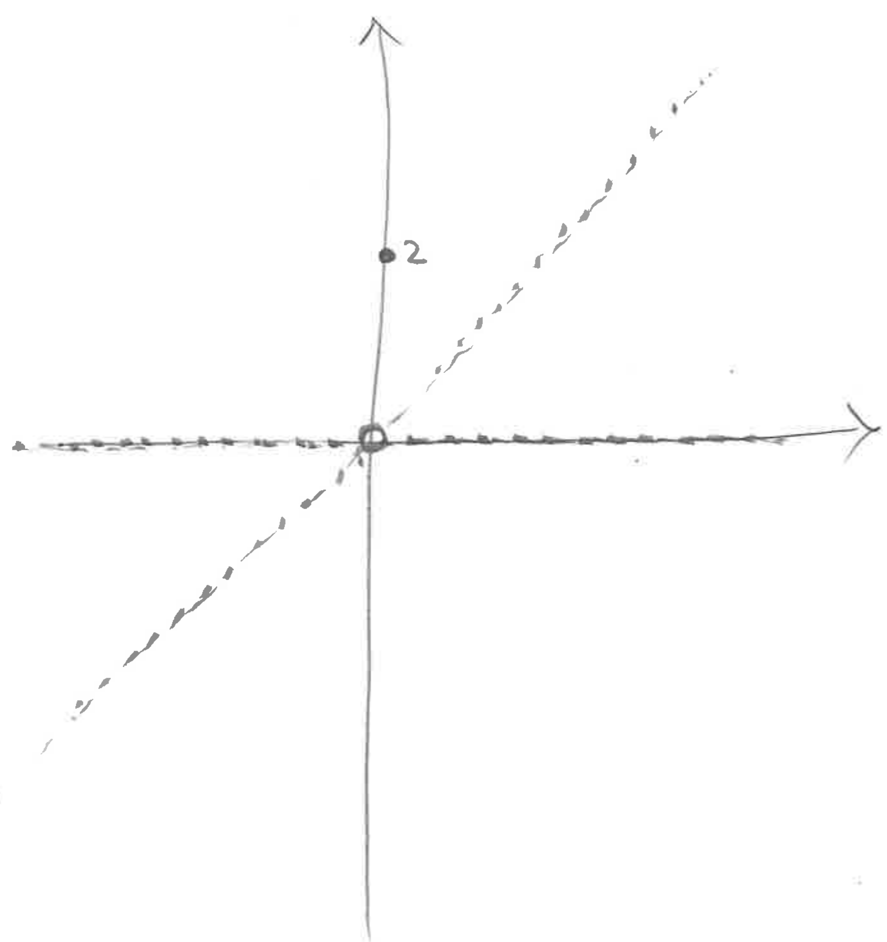
$$|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$$ 3 marks

$$|x| < \delta \Rightarrow |f(x)| < \varepsilon$$

Since $|f(x)| \leq |x|$ for all x it is enough to choose $\delta = \varepsilon$. 4 marks

3. (c) **(5 marks)** Give an example of a function f defined on the set of real numbers $\mathbb{R}$ which has jump discontinuity for all $x \neq 0$ and removable discontinuity at $x = 0$.

$$f(x) = \begin{cases} 2 & x = 0 \\ 0 & x \text{ is an irrational number} \\ x & x \text{ is a non-zero rational} \end{cases}$$



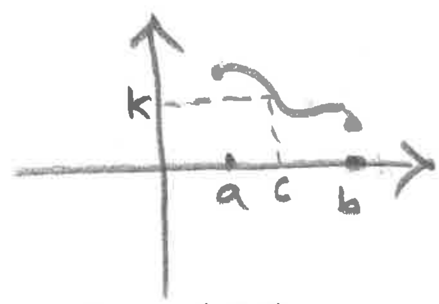
f has jump discontinuity at all points $x \neq 0$ but at $x = 0$ it has removable discontinuity. (we could remove discontinuity by defining $f(0) = 0$)

4 (15 marks)

4.(a) **(4 marks)** State the Intermediate Value Theorem (IVT).

F is a continuous function on the closed interval $[a,b]$. Let k be a number between $F(a)$ and $F(b)$. Then there is a number $c \in (a,b)$ such that

$$F(c) = k$$



4.(b) **(4 marks)** Use IVT to show that there is some number $x \in (-2, 0)$ such that

$$x^3 + \frac{1}{2} = 2\sin x.$$

Let $F(x) = x^3 + \frac{1}{2} - 2\sin x$

F continuous on $[-2,0]$ — 1 mark

$F(-2) = (-2)^3 + \frac{1}{2} - 2\sin(-2)$

$= -8 + \frac{1}{2} + 2\sin(2)$

$= (-7.5) + 2\sin(2) < 0$ — 1 mark

$-1 \leq \sin x \leq 1 \Rightarrow 2\sin(2) \leq 2$

$F(0) = 0 + \frac{1}{2} - 2\sin 0 = \frac{1}{2} > 0$ — 1 mark

By IVT there must be a point $c \in (-2,0)$ such that $F(c) = 0$

$c^3 + \frac{1}{2} - 2\sin c = 0$ — 1 mark

$c^3 + \frac{1}{2} = -2\sin c$

4. c) **(7 marks)** Show that the cubic equation

$$x^3 + ax^2 + bx + c = 0$$

has at least one real root. Here a, b and c are fixed real numbers.

1 mark

Define $f(x) = x^3 + ax^2 + bx + c$ on $\mathbb{R}$. f continuous every where.

$$\lim_{x \to -\infty} (x^3 + ax^2 + bx + c) = \lim_{x \to -\infty} x^3 \left(1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}\right)$$

$$= \lim_{x \to -\infty} x^3 \cdot (1 + 0 + 0 + 0)$$

$$= \lim_{x \to -\infty} x^3 = -\infty$$

Thus, there must be a number "m" such that

$$f(m) = m^3 + am^2 + bm + c < 0$$

3 marks

Similarly, $\lim_{x \to +\infty} (x^3 + ax^2 + bx + c) = +\infty$

So there must be a number "n" such that

$$f(n) = n^3 + an^2 + bn + c > 0$$

2 marks

Thus, by IVT there is a number p between m and n such that

$$f(p) = p^3 + ap^2 + bp + c = 0$$

1 mark

5 **(20 marks)**

5.(a) **(5 marks)** Use definition of derivative to show that $g'(0) = 0$ if

$$g(x) = x \sin x.$$

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$$ 1 mark

$$= \lim_{h \to 0} \frac{h\,g(h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h \sin h}{h}$$ 2 mark

$$= \lim_{h \to 0} (\sin h) = \sin 0 = 0$$ 2 mark

Note: $y = \sin x$ a continuous function,

So $\lim_{x \to 0} \sin x = \sin 0 = 0$

5.(b) **(15 marks)** Calculate y'.

i. $y = (a + \frac{b}{x^2})^3$, here a and b are fixed numbers.

ii. $y = e^{\sqrt{\frac{x+1}{x-1}}}$

iii. $e^y \cos x = 1 + \sin(xy)$

iv. $y = \ln(\sin(\cos x))$

v. $y = x^{\arctan x}$

i

$$y' = 3\left(a + \frac{b}{x^2}\right)^2 \cdot \frac{d}{dx}\left(a + \frac{b}{x^2}\right)$$

$$= 3\left(a + \frac{b}{x^2}\right)^2 \cdot \frac{-2b}{x^3}$$

$$= -\frac{6b}{x^3}\left(a + \frac{b}{x^2}\right)^2$$

3 marks

ii

$$y' = e^{\sqrt{\frac{x+1}{x-1}}} \cdot \frac{d}{dx}\left(\sqrt{\frac{x+1}{x-1}}\right)$$

$$= e^{\sqrt{\frac{x+1}{x-1}}} \cdot \frac{1}{2} \cdot \sqrt{\frac{x-1}{x+1}} \cdot \frac{(1)(x-1)-(1)(x+1)}{(x-1)^2}$$

$$= \frac{1}{2}\sqrt{\frac{x-1}{x+1}} \cdot \frac{-2}{(x-1)^2} \cdot e^{\sqrt{\frac{x+1}{x-1}}}$$

$$= \frac{-1}{\sqrt{(x+1)}\sqrt{(x-1)^3}} \cdot e^{\sqrt{\frac{x+1}{x-1}}}$$

3 marks

iii

$$y' e^y \cos x + e^y(-\sin x) = \cos(xy) \cdot (y + xy')$$

1 mark

1 mark

$$y' e^y \cos x - x\cos(xy)\, y' = e^y \sin x + y\cos(xy)$$

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$$y' = \frac{e^y \sin x + y\cos(xy)}{e^y \cos x - x\cos(xy)}$$

1 mark

(iv) $$y' = \frac{1}{\sin(\cos x)} \cdot \frac{d}{dx}(\sin(\cos x))$$

$$= \frac{1}{\sin(\cos x)} \cdot \cos x(\cos x) \cdot \frac{d}{dx}(\cos x)$$

$$= \frac{-\sin x \, \cos(\cos x)}{\sin(\cos x)} = -\sin x \cdot \cot(\cos x)$$

3 marks

(v) $$y = x^{\arctan x}$$

$$\ln y = \ln x^{\arctan x}$$

1 mark

$$= (\arctan x)\ln x$$

$$\frac{y'}{y} = \frac{1}{(1+x^2)} \cdot \ln x + \frac{\arctan x}{x}$$

1 mark

$$y' = y \cdot \left[\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right]$$

$$= x^{\arctan x}\left[\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right]$$

1 mark

6 **(10 marks)**

6. **(10 marks)** Sketch the graph of a function which satisfies all of the given conditions:

$f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $1 < |x| < 2$

$f'(x) = -1$ if $|x| > 2$, $f''(x) < 0$ if $-2 < x < 0$, inflection point $(0, 1)$.

$f'(x) < 0$ if $|x| < 1$. So

$f'(x) < 0$ if $x \in (-1, 1)$.

Thus f decreasing on $(-1, 1)$

1 mark

$f'(x) > 0$ if $1 < |x| < 2$ So

$f'(x) > 0$ if $x \in (1, 2) \cup (-2, -1)$

Thus f increasing when $x \in (1, 2) \cup (-2, -1)$

1 mark

$f'(x) = -1$ if $|x| > 2$

$f'(x) = -1$ if $x \in (-\infty, 2) \cup (2, +\infty)$

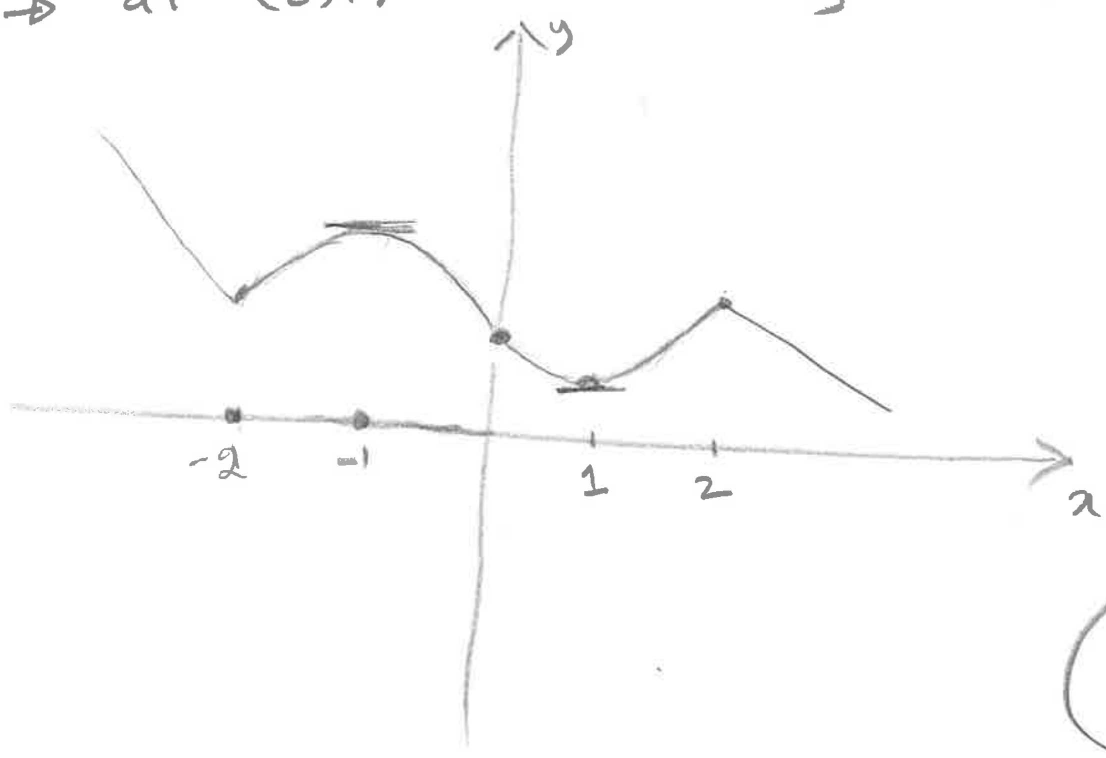
1 mark

The graph of f has constant slope -1.

on $(-2, 0)$ the function is concave down

at $(0, 1)$ the concavity changes.

1 mark



6 marks

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