Term practice test

Q1 (a) $\text{Dom}(f) = ?$ $\quad f(x) = \sqrt{\dfrac{x^3-9x}{x^2-1}} + \dfrac{\ln x}{\sqrt{x^2+1}}$

Ans: $\begin{cases} x > 0 \\ \dfrac{x^3-9x}{x^2-1} \geq 0 \end{cases} \Rightarrow P = \dfrac{x^3-9x}{x^2-1} = \dfrac{x(x-3)(x+3)}{(x-1)(x+1)}$

Use the following table

x		-3		-1		0		1		3	
x	$-$		$-$		$-$	0	$+$		$+$		$+$
$x-3$	$-$		$-$		$-$		$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$		$+$		$+$		$+$
$x-1$	$-$		$-$		$-$		$-$	0	$+$		$+$
$x+1$	$-$		$-$	0	$+$		$+$		$+$		$+$
P	$-$	0	$+$	ND	$-$	0	$+$	ND	$-$	0	$+$

$\text{Dom} = (0, 1) \cup [3, +\infty)$

Q1.(b)

(a) $f(x) = x^2 + \dfrac{1}{x^3}$: neither even nor odd

$\begin{cases} f(1) = 2 \\ f(-1) = 0 \end{cases}$ & $f(1) \neq f(-1)$ and $f(1) \neq -f(-1)$.

(b) $g(x) = x\left(\dfrac{\cos x}{|x|} + e^{-x^2}\right)$

$g(-x) = -x\left(\dfrac{\cos(-x)}{|-x|} + e^{-(-x)^2}\right)$

$= -x\left(\dfrac{\cos x}{|x|} + e^{-x^2}\right) = -g(x)$.

note that $\text{Dom}\, g = \mathbb{R} - \{0\}$

g is even.

1. (c) $\ln(\ln x) = 1 \Rightarrow e^{\ln(\ln x)} = e \Rightarrow$

$\ln x = e \Rightarrow e^{\ln x} = e^{e} \Rightarrow \boxed{x = e^{e}}$

2. (a) $\lim_{x \to -1} (e^{k}x + 1) = -e^{k} + 1$.

$\forall \varepsilon > 0, \exists \delta > 0 \;; 0 < |x+1| < \delta \Rightarrow |e^{k}x + 1 + e^{k} - 1| < \varepsilon$

Given $\varepsilon > 0$: $|e^{k}x + e^{k}| < \varepsilon \Rightarrow |e^{k}(x+1)| < \varepsilon$

$\Rightarrow e^{k}|x+1| < \varepsilon \Rightarrow |x+1| < \frac{\varepsilon}{e^{k}} = e^{-k} \cdot \varepsilon$

enough to $\delta \leq e^{-k} \cdot \varepsilon$, since: $|x+1| < \delta = e^{-k} \cdot \varepsilon$

$\rightarrow |e^{k}(x+1)| < \varepsilon \Rightarrow |e^{k}x + e^{k}| < \varepsilon \Rightarrow |e^{k}x + 1 + e^{k} - 1| < \varepsilon$

$\Rightarrow |f(x) - (-e^{k} + 1)| < \varepsilon$. ✓

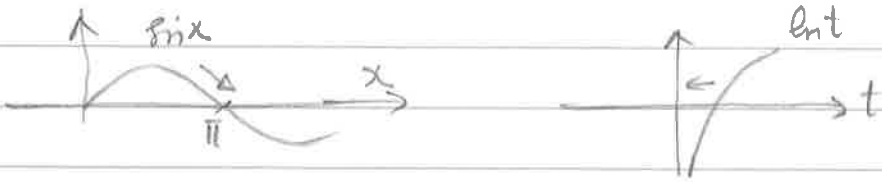
2. (b) (1) $\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{x \to -3} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}(x-1)} = \frac{-6}{-4} = \frac{3}{2}$

(2) $\lim_{x \to 3} \frac{\sqrt{x+6} - x}{x^2(x-3)} \cdot \frac{\sqrt{x+6} + x}{\sqrt{x+6} + x} =$

$\lim_{x \to 3} \frac{x + 6 - x^2}{x^2(x-3)(\sqrt{x+6} + x)} = \lim_{x \to 3} \frac{-\cancel{(x-3)}(x+2)}{x^2\cancel{(x-3)}(\sqrt{x+6} + x)}$

$= \frac{-5}{54}$.

(3) $\lim\limits_{x\to\pi^-} \ln(\sin x) = \lim\limits_{t\to 0^+} \ln(t) = -\infty.$



(4) $\lim\limits_{x\to\infty} e^{-x^2+x} = \lim\limits_{t\to-\infty} e^{t} = 0$

$t = -x^2+x$ note that $(x\to\infty \Rightarrow t\to-\infty)$.

(5) $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to\infty} \dfrac{2\sqrt{x}}{x} = \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$

Q3. (a) $\lim\limits_{x\to3^+} f(x) = \lim\limits_{x\to3^-} f(x) = f(3)$

$$\begin{cases} \lim\limits_{x\to3^-} (cx^2+5x) = 9c+15 \\ \lim\limits_{x\to3^+} (x^3-cx) = 27-3c \\ f(3) = 27-3c \end{cases}$$

$\Rightarrow \quad 9c+15 = 27-3c \Rightarrow 12c=12 \Rightarrow \boxed{c=1}.$

Q3. (b) f has to be a constant function.

Check Assignment 3 solution.

Q3. (c) $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

this is continuous only at $x=0$.

(Look at assignment 2).

Q4. (a) : Suppose f is continuous on $[a,b]$ and $f(a)\cdot f(b) < 0$, then there exists a $c \in (a,b)$ such that $f(c)=0$.

Q4. (b) Let $f(x) = 10x^3 + 5x^2 - 3e^x + 2$.

f is continuous on $[0,1]$,

$f(0) = -1 < 0$, $f(1) = 15 - 3e + 2 = 17 - 3e > 0$

So, there exists $x \in (0,1)$, such that $f(x) = 0$.

that is, $10x^3 + 5x^2 - 3e^x + 2 = 0$ OR

$\exists\, x \in (0,1)$: $10x^3 + 5x^2 = 3e^x - 2$.

Q4. (c) Let $g(x) = f(x) - x$ then g is continuous on $[a,b]$ since f is so.

Suppose $f(0) \neq 0$, otherwise 0 is the point that we're looking for. $\Rightarrow f(0) > 0$.

and suppose $f(1) \neq 1 \Rightarrow f(1) < 1$.

Now, $g(0) = f(0) > 0$, $g(1) = f(1) - 1 < 0$, So by IVT $\exists\, x \in (0,1)$ such that $g(x) = 0$ OR $f(x) = x$.

5. (a)

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x^{3/7}}{x}$$

$$= \lim_{x \to 0} \frac{1}{x^{4/7}} = \lim_{x \to 0} \frac{1}{\sqrt[7]{x^4}} = \infty.$$

So, $g'(0)$ is not a real #.

Q 5. (b) ① $y' = \frac{(x^2 \ln x)'}{x^2 \ln x} = \frac{2x \ln x + x^2 \frac{1}{x}}{x^2 \ln x}$

$$= \frac{2x \ln x + x}{x^2 \ln x}.$$

② $y' = -\sin x \, e^{\cos x} - e^x \cdot \sin(e^x).$

③ $(xy)' \cos(xy) = 2x - y'$

$(y + xy') \cos(xy) = 2x - y' \Rightarrow$

$y \cos(xy) + xy' \cos(xy) = 2x - y' \Rightarrow$

$xy' \cos(xy) + y' = 2x - y \cos(xy)$

$y' [x \cos(xy) + 1] = 2x - y \cos(xy)$

$$y' = \frac{2x - y \cos(xy)}{x \cos(xy) + 1}.$$

(4) $y = (\tan x)^{1/x} \Rightarrow y' = ?$

$$\ln y = \ln(\tan x)^{1/x} = \tfrac{1}{x}\ln(\tan x)$$

$$\Rightarrow \frac{y'}{y} = \left(-\frac{1}{x^2}\right)\ln(\tan x) + \frac{\sec^2 x}{\tan x}\cdot\frac{1}{x}$$

$$\Rightarrow y' = y\left\{-\frac{\ln(\tan x)}{x^2} + \frac{\sec^2 x}{x\tan x}\right\}$$

$$\Rightarrow y' = (\tan x)^{1/x}\cdot\left\{-\frac{\ln(\tan x)}{x^2} + \frac{\sec^2 x}{x\tan x}\right\}.$$

(5) $$y' = \frac{\left[\sqrt{\frac{1-x}{1+x}}\right]'}{1+\left(\frac{1-x}{1+x}\right)} = \frac{\dfrac{-2/(1+x)^2}{2\left(\sqrt{\frac{1-x}{1+x}}\right)}}{1+\left(\frac{1-x}{1+x}\right)}$$

$$= \frac{-1/(1+x)^2\sqrt{\frac{1-x}{1+x}}}{1+\left(\frac{1-x}{1+x}\right)} = \frac{-(1+x)}{2(1+x)^2\left(\sqrt{\frac{1-x}{1+x}}\right)}.$$

$$= \frac{-1}{2(1+x)\sqrt{\frac{1-x}{1+x}}}$$

Q6.(a) by IVT, $f(x) = 3x + 2\cos x + 5$ has one root between $(-\pi, 0)$, since,

$$f(-\pi) = -3\pi + 2\cos(-\pi) + 5$$
$$= -3\pi - 2 + 5 < 0$$

$f(0) = 2\cos 0 + 5 = 7 > 0$, $\exists x_1 \in (-\pi, 0) : f(x_1) = 0$.

but if there is another root like x_2,

then since $f(x_1) = f(x_2) = 0$ and f is differentiable $\Rightarrow$ by Rolle's thm, $\exists\ x_3 \in (x_1, x_2)$ such that $f'(x_3) = 0$, but $f'(x) = 0$ has no solution since

$f'(x) = 3 - 2\sin x = 0 \qquad \sin x = \frac{3}{2}$ no solution.

This contradiction says there is no other root.

Q 6 (b) $\quad y = e^{-x^2} \qquad Dom = (-\infty, +\infty)$

$\lim\limits_{x \to \pm\infty} y = \lim\limits_{x \to \pm\infty} e^{-x^2} = 0 \qquad y = 0$ is horizontal asymptote.

$y' = -2x e^{-x^2} \Rightarrow y' = 0 \Rightarrow -2x = 0 \Rightarrow \boxed{x = 0}$ critical pt

$y'' = -2e^{-x^2} + 4x^2 e^{-x^2}$

$= -2(1 - 2x^2) e^{-x^2} = 0 \Rightarrow 1 - 2x^2 = 0$

$\rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{\sqrt{2}}{2}$ inflection pt.

x	$-\infty$		$-\frac{\sqrt{2}}{2}$		0		$+\frac{\sqrt{2}}{2}$		$+\infty$
y'		$+$		$+$	0	$-$		$-$	
y''		$+$	0	$-$		$-$	0	$+$	
y		$\nearrow$		$\nearrow$		$\searrow$		$\searrow$	
					local max