York University

Faculty of Science and Engineering
MATH 2030 3.00MW – Elementary Probability
Instructor: T. Salisbury

Second Midterm Examination – Solutions
March 18, 2011

NAME: ______________________________

SIGNATURE: ______________________________

STUDENT NUMBER:______________________________

Instructions:

(1) You have 50 minutes to complete this exam.
(2) There are 4 questions on 6 pages, worth a total of 100 points. If you run out of room, use the back of the page.
(3) You may use calculators and may refer to a formula sheet (a letter-size sheet with writing on either side). You may not refer to other notes, books, or use wireless internet devices.
(4) Show your work, and explain or justify your solutions if possible. You may leave numerical answers unsimplified.
(5) A table of the normal cdf is included. When using it, you may use the closest table value rather than interpolating.

1. A random variable X has density function

$$f(x) = \begin{cases} 0, & x < 0 \\ c + x, & 0 < x < 1 \\ 0, & x > 1. \end{cases}$$

(a) [5] Find c.
(b) [10] Find the cumulative distribution function of X.
(c) [10] Find $E[X]$.

Solution:

(a) $1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 (c + x)\,dx = \Big[cx + \frac{x^2}{2}\Big]_0^1 = c + \frac{1}{2}$. So $c = \frac{1}{2}$.

(b) $F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} \int_{-\infty}^{x} 0\,dt = 0, & x < 0 \\ \int_{-\infty}^{0} 0\,dt + \int_0^x (\frac{1}{2} + t)\,dt = \left[\frac{t}{2} + \frac{t^2}{2}\right]_0^x = \frac{x+x^2}{2}, & 0 \le x < 1 \\ \int_{-\infty}^{0} 0\,dt + \int_0^1 (\frac{1}{2} + t)\,dt + \int_1^x 0\,dt = 1, & 0 \le x < 1. \end{cases}$

(c) $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(\frac{1}{2} + x)\,dx = \left[\frac{x^2}{4} + \frac{x^3}{3}\right]_0^1 = \frac{7}{12}.$

2. A discrete random variable X has distribution

x	-1	0	5	7
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $Y \sim N(3, 7)$ and let A be the event that $X \ge 2$.

(a) [10] Find $E[X + 2Y - 1_A + 5]$.
(b) [10] Find $E[X 1_A]$.

Solution:

(a) $E[X] = -1 \times \frac{1}{2} + 0 \times \frac{1}{6} + 5 \times \frac{1}{6} + 7 \times \frac{1}{6} = \frac{9}{6}$. Also $E[Y] = 3$ and $E[1_A] = P(A) = P(X = 5 \text{ or } X = 7) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$. So $E[X + 2Y - 1_A + 5] = E[X] + 2E[Y] - E[1_A] + 5 = \frac{73}{6}$.

(b) The possible values of $X 1_A$ are 0 (probability $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$), 5 (probability $\frac{1}{6}$), and 7 (probability $\frac{1}{6}$). So $E[X 1_A] = 0 \times \frac{2}{3} + 5 \times \frac{1}{6} + 7 \times \frac{1}{6} = \frac{12}{6} = 2$.

[Or, though we didn't study the formula $E[g(X)] = \sum g(x) P(X = x)$ till after the midterm cutoff, I would also have accepted $E[X 1_A] = -1 \times 0 \times \frac{1}{2} + 0 \times 0 \times \frac{1}{6} + 5 \times 1 \times \frac{1}{6} + 7 \times 1 \times \frac{1}{6} = 2$.]

3. Let $Z \sim N(0, 1)$, and $Y = Z^{1/3}$.

(a) [15] Find the density of Y.
(b) [10] Find $P\Big(Y \ge (\frac{1}{2})^{1/3}\Big)$.

Solution:

(a) Let $H(y)$ be the cdf of Y. Then since the cubed-root function is 1-1, $H(y) = P(Y \le y) = P(Z^{1/3} \le y) = P(Z \le y^3) = \Phi(y^3)$. Therefore the density of Y is $H'(y) = 3y^2 \Phi'(y^3) = \frac{3y^2}{\sqrt{2\pi}} e^{-y^6/2}$.

(b) $P(Y \ge (1/2)^{1/3}) = 1 - H((1/2)^{1/3}) = 1 - \Phi(\frac{1}{2}) = 1 - 0.6915 = 0.3085$
[Or alternately, $P(Y \ge (1/2)^{1/3}) = P(Z \ge \frac{1}{2}) = \ldots$]

4. Ralph enrols in Calculus by mistake, and only discovers this after the end of classes. He decides to show up to the exam anyway. The exam consists of 50 multiple-choice questions, worth 1 mark each, with possible answers (a) (b) (c) (d) (e). He has no clue how to answer any of the questions, so fills in answers guessed at random.

(a) [5] What is his mean score? (You must justify your answer to receive credit)

(b) [25] What (approximately) is the probability that he gets 15 or more questions correct?

Solution:

(a) Let X be his score. Then $X \sim \text{Bin}(50, \frac{1}{5})$, so $E[X] = np = 50/5 = 10$.

(b) $np(1-p) = 8$ so $P(X \geq 15) = P(X \geq 14.5) = P\Big(\frac{X-10}{\sqrt{8}} \geq \frac{14.5-10}{\sqrt{8}}\Big)$
$\approx P(Z \geq 1.5910) \approx P(Z \geq 1.59) = 1 - \Phi(1.59) = 1 - 0.9441 = 0.0559$