University of Toronto
Department of Mathematics

# MAT223H1F

Linear Algebra I

## Midterm Examination

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Duration: 1 hour 20 minutes

Last Name: ______________________________

Given Name: ______________________________

Student Number: ______________________________

Tutorial Code: ______________________________

**No calculators or other aids are allowed.**

| FOR MARKER USE ONLY | |
|---|---|
| Question | Mark |
| 1 | /10 |
| 2 | /10 |
| 3 | /10 |
| 4 | /10 |
| 5 | /5 |
| 6 | /5 |
| TOTAL | /50 |

**1.** Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 + x_5 &= 2 \\ 2x_1 + x_2 + 3x_3 + 5x_4 + 5x_5 &= 7 \\ 3x_1 + 6x_2 + 4x_3 + 9x_4 + 10x_5 &= 11 \\ x_1 + 2x_2 + 4x_3 + 3x_4 + 6x_5 &= 9. \end{aligned}$$

Find all solutions to the system and express your answer in parametric form.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 5 & 5 & 7 \\ 3 & 6 & 4 & 9 & 10 & 11 \\ 1 & 2 & 4 & 3 & 6 & 9 \end{array}\right) \xrightarrow{R_2=R_2-2R_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & 1 & 3 & 3 \\ 3 & 6 & 4 & 9 & 10 & 11 \\ 1 & 2 & 4 & 3 & 6 & 9 \end{array}\right)$$

$$\xrightarrow{R_3=R_3-3R_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 7 & 5 \\ 1 & 2 & 4 & 3 & 6 & 9 \end{array}\right) \xrightarrow{R_4=R_4-R_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 3 & 1 & 5 & 7 \end{array}\right)$$

$$\xrightarrow{R_4=R_4-3R_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 0 & -8 & -16 & -8 \end{array}\right) \xrightarrow{R_4=-1/8R_4} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & -3 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{R_2=-1/3R_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & -1/3 & -1/3 & -1 & -1 \\ 0 & 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right)$$

Since we have four leading ones and five variables, we will have one parameter. Let $x_5 = t$, where $t$ is any real number. The fourth row implies that

$$x_4 + 2x_5 = 1 \Rightarrow x_4 = 1 - 2t.$$

The third row implies that

$$x_3 + 3x_4 + 7x_5 = 5 \Rightarrow x_3 + 3(1-2t) + 7t = x_3 + 3 + t = 5 \Rightarrow x_3 = 2 - t.$$

The second row implies that

$$x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 - x_5 = -1 \Rightarrow x_2 - \frac{1}{3}(2-t) - \frac{1}{3}(1-2t) - t = x_2 - 1 = -1 \Rightarrow x_2 = 0.$$

The first row implies that

$$x_1 + 2x_2 + x_3 + 2x_4 + x_5 = 2 \Rightarrow x_1 + (2-t) + 2(1-2t) + t = 2 \Rightarrow x_1 = -2 + 4t.$$

Hence, the general solution is

$$x_1 = 4t - 2, x_2 = 0, x_3 = 2 - t, x_4 = 1 - 2t, x_5 = t,$$

where $t$ is any real number.

EXTRA PAGE FOR QUESTION 1 - do not remove.

**2.** Suppose that $A$ is a $3 \times 3$ matrix, $A^T + 5I_3$ is nonsingular (invertible) and

$$(A^T + 5I_3)^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix}.$$

Find $A$.

First, find $\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix}^{-1}$ using Gaussian elimination:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -4 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 = -R_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -4 & 1 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 + 4R_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right)$$

Hence, we have

$$A^T + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ -2 & -4 & 1 \end{pmatrix}$$

$$\Rightarrow A^T = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ -2 & -4 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} -4 & 3 & 0 \\ 0 & -6 & 0 \\ -2 & -4 & -4 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} -4 & 3 & 0 \\ 0 & -6 & 0 \\ -2 & -4 & -4 \end{pmatrix}^T = \begin{pmatrix} -4 & 0 & -2 \\ 3 & -6 & -4 \\ 0 & -4 & -4 \end{pmatrix}$$

EXTRA PAGE FOR QUESTION 2 - do not remove.

**3.** Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

**(a)** Find elementary matrices $E_1, E_2$ and $E_3$ such that $E_3E_2E_1A = I_3$.

**(b)** Write $A$ as a product of elementary matrices.

(a): Reduce $A$ to $I_3$:

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2=R_2-3R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2=1/4R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1=R_1+2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, $E_3E_2E_1A = I_3$, where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b): From (a), we have that

$$A = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

EXTRA PAGE FOR QUESTION 3 - do not remove.

**4.** In each case below, determine whether $W$ is a subspace of the vector space $V$. Justify your answer.

**(i)** Let $V = P_2$ be the vector space of all polynomials of degree less than or equal to 2. Let $W = \{ax^2 + bx + c \mid a^2 = c\}$.

**(ii)** Let $V = M_{nn}$ be the vector space of all $n \times n$ matrices. Let $W = \{A \in M_{nn} \mid AB = BA\}$ where $B \in M_{nn}$ is fixed.

(i): $W$ is not closed under addition: $x^2 + 1 \in W$, but $(x^2 + 1) + (x^2 + 1) = 2x^2 + 2 \notin W$. Therefore, $W$ is not a subspace of $V$.

(ii): Fix any $B \in M_{nn}$.

(0): Since $I_n B = B I_n = B$, the identity $I_n$ is an element of $W$, so $W$ is non-empty.

(1): If $A, A' \in W$ then $AB = BA$ and $A'B = BA'$, so

$$(A + A')B = AB + A'B = BA + BA' = B(A + A').$$

Hence, $A + A' \in W$.

(2): If $A \in W$, we have $AB = BA$. Then, for any $c \in \mathbb{R}$,

$$(cA)B = c(AB) = c(BA) = B(cA),$$

so $cA \in W$.

This shows that $W$ is a subspace of $V$.

**5.** Let $V = \{A \in M_{22} \mid A^{-1} \text{ exists}\}$. Define the operation of vector addition in $V$ by

$$A + B = AB^{-1}$$

and let scalar multiplication in $V$ be the usual scalar multiplication in $M_{22}$. Show that $V$ is not a vector space by demonstrating with an example that one of the properties in the definition of a vector space fails to hold.

Most properties of vector spaces fail to hold for this example. For example, $V$ is not closed under scalar multiplication: $I_2$ is invertible and therefore an element of $V$, but

$$0 \cdot I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is not.

Another property which fails to hold is commutativity of "+":

$$I_2 + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix},$$

while

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + I_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} I_2^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

**6.** Suppose that $W$ is a non-empty subset of a vector space $V$. Suppose that $W$ satisfies the following two conditions:

(i) For every $u$ and $v \in W$, $2u - v \in W$.

(ii) For every $u \in W$, and every real number $c$, $cu \in W$.

Prove that $W$ is a subspace of $V$.

It is enough to show that $W$ satisfies the conditions of the "Subspace Theorem":

(0) $W$ is non-empty.

(1) If $u, v \in W$ then $u + v \in W$.

(2) If $u \in W$ and $c \in \mathbb{R}$ then $cu \in W$.

We will show these hold.

(0) $W$ was assumed to be non-empty.

(2) This follows from assumption (ii).

(1) Let $u$ and $v$ be any elements of $W$. By (ii), $1/2u \in W$ and $-v \in W$, so (i) implies that

$$2(\frac{1}{2}u) - (-v) \in W \Rightarrow u + v \in W.$$

Hence, $W$ is a subspace of $V$.