MATH 3090 Computational Mathematics Fall 2010

Mid-term exam 1/ October 20, 2010

Student Name: **ID-No.:** Solutions

You have 50 minutes to solve the following problems: Show your complete work. A prewritten help-sheet is allowed: No other aids (such as laptop computer) are permitted. The maximal score on this test is 100.

Problem 1 **(30 marks)**

Consider the following linear system:

$$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \\ 12 \end{pmatrix}$$

Solve the system approximatively by using the Gauss-Seidel algorithm, beginning with

$x^{(0)} = (1, 1, 1)^T$.

Gauss-Seidel algorithm:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \cdot \left(b_i - \sum_{j=1}^{i-1} a_{ij} \cdot x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} \cdot x^{(k)} \right)$$

$x^{(0)} = (1, 1, 1)$

$k = 1: \quad x_1^{(1)} = \frac{1}{5} \cdot (10 - 1 \cdot 1 - 1 \cdot 1) = \frac{8}{5} = 1.6$

$x_2^{(1)} = \frac{1}{4} \cdot (12 - 1 \cdot 1.6 - 1 \cdot 1) = \frac{9.4}{4} = 2.35$

$x_3^{(1)} = \frac{1}{3} \cdot (12 - 1 \cdot 1.6 - 1 \cdot 2.35) = 2.683$

$k = 2: \quad x_1^{(2)} = \frac{1}{5} \cdot (10 - 1 \cdot 2.35 - 1 \cdot 2.683)$

$= 0.9934$

$x_2^{(2)} = \frac{1}{4} \cdot (12 - 1 \cdot 0.9934 - 1 \cdot 2.683)$

$= 2.0809$

$x_3^{(2)} = \frac{1}{3} \cdot (12 - 1 \cdot 0.9934 - 1 \cdot 2.0809)$

$= 2.9752$

$$\Rightarrow \quad x^{(1)} = \begin{pmatrix} 1.6 \\ 2.35 \\ 2.683 \end{pmatrix} \quad ; \quad x^{(2)} = \begin{pmatrix} 0.9934 \\ 2.0809 \\ 2.9752 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(solution)

Problem 2 **(40 marks)**

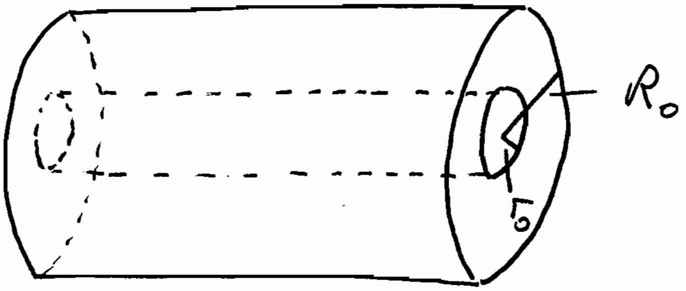
Consider the steady state axially symmetric heat conduction problem

$0 = rf + (Kru_r)_r,$

$u(r_0)$ = given and $u(R_0)$ = given, where $0 < r_0 < R_0$:

Here f=f(r) is a heat source term.

Find a discrete model and the solution to the resulting algebraic problem, by choosing n=4 equidistant points between r_0 and R_0 .

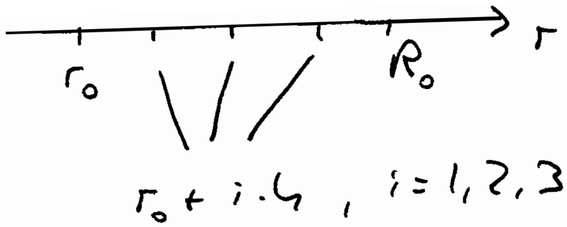


Use the discrete approximations:

$$\frac{\partial u}{\partial r} \approx \frac{u_{i+1} - u_i}{h}$$

$$\frac{\partial^2 u}{\partial r^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

$$f_i = f(i \cdot h + r_0), \quad \text{where} \quad h = \frac{R_0 - r_0}{4}$$



discrete approximation:

$$0 = r\cdot f + \frac{\partial}{\partial r}\left(K\cdot r\cdot\frac{\partial u}{\partial r}\right)$$

$$= r\cdot f + K\cdot\frac{\partial u}{\partial r} + K\cdot\frac{\partial^2 u}{\partial r^2}$$

$i=1$:

$$0 = (r_0+h)\cdot f_1 + K\cdot\frac{u_1 - u(r_0)}{h} + K\cdot(r_0+h)\cdot\frac{u_2 - 2u_1 + u(r_0)}{h^2}$$

$i=2$:

$$0 = (r_0+2h)\cdot f_2 + K\cdot\frac{u_2 - u_1}{h} + K\cdot(r_0+2h)\cdot\frac{u_3 - 2u_2 + u_1}{h^2}$$

$i=3$:

$$0 = (r_0 + 3h)\cdot f_3 + K\cdot\frac{u_3 - u_2}{h} + K\cdot(r_0+3h)\cdot\frac{u(R_0) - 2u_3 + u_2}{h^2}$$

This system can be brought into the form

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

where $\tilde{f}_1$ and $\tilde{f}_3$ contain boundary terms:

The system can be solved by using the tridiagonal algorithm.

Problem 3 **(30 marks)**

Consider the linear problem $x = Ax + d$ with the following matrix A and vector d:

$$A = \begin{pmatrix} 0 & 0.3 & -0.4 \\ 0.4 & 0 & 0.2 \\ 0.3 & 0.1 & 0 \end{pmatrix}, \qquad d = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}.$$

a) Verify that the norm of A is less than 1.

b) State a general formula for the inverse of I – A, where I is the 3x3 identity matrix.

c) Calculate an approximate solution x from the general formula in part b) (using powers A^0, A^1, A^2), or from a suitable iteration $x^{(k)}$, beginning with $x^{(0)} = (0, 0, 0)$.

a) $$\|A\| = \max_i \sum_{j=1}^{3} |a_{ij}| = \max\{0.7, 0.6, 0.4\}$$
$$= 0.7$$

b) $$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

$$x = Ax + d \iff (I - A)\cdot x = d$$
$$\Rightarrow x = (I - A)^{-1} d$$
$$\Rightarrow x = \left(\sum_{k=0}^{\infty} A^k\right)\cdot d$$

$$A^2 = \begin{pmatrix} 0 & 0.3 & -0.4 \\ 0.4 & 0 & 0.2 \\ 0.3 & 0.1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0.3 & -0.4 \\ 0.4 & 0 & 0.2 \\ 0.3 & 0.1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -0.04 & 0.06 \\ 0.06 & 0.14 & -0.16 \\ 0.04 & 0.09 & -0.10 \end{pmatrix}$$

approximate solution:

$$x = (I + A + A^2) \cdot d$$

$$= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0.3 & -0.4 \\ 0.4 & 0 & 0.2 \\ 0.3 & 0.1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -0.04 & 0.06 \\ 0.06 & 0.14 & -0.16 \\ 0.04 & 0.09 & -0.10 \end{pmatrix} \right] \cdot \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0.26 & -0.34 \\ 0.46 & 1.14 & 0.04 \\ 0.34 & 0.19 & 0.90 \end{pmatrix} \cdot \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.46 \\ 0.82 \\ 0.715 \end{pmatrix}$$

check the approximate solution:

$$x - Ax = \begin{pmatrix} 0.46 \\ 0.82 \\ 0.715 \end{pmatrix} - \begin{pmatrix} 0 & 0.3 & -0.4 \\ 0.4 & 0 & 0.2 \\ 0.3 & 0.1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0.46 \\ 0.82 \\ 0.715 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5 \\ 0.493 \\ 0.495 \end{pmatrix} \approx d .$$