MAT 1332 W2011 TEST1

Max = 30

Solutions

1. Area calculation. **(Total=3)** Marking: 1 point for sketch, 1 point for correctly setting up the integral including correct limits of integration, 0.5 point for correct ant derivative. 0.5 points for correct end result (can be in exact form).

1. Let R be the region bounded by the curves $y = 0.5x(x+4)$ and $y = -x(x+4)$.

   (a) Sketch

   (b) We have $0.5x(x+4) = -x(x+4) \iff x(x+4) = 0 \iff x = 0$ or $x = -4$. Moreover the curve $y = 0.5x(x+4)$ lies below the curve $y = -x(x+4)$ for $-4 < x < 0$. We conclude that the area in question is

$$A = \int_{-4}^{0} -x(x+4) - 0.5x(x+4)dx = -1.5\int_{-4}^{0} x(x+4)dx$$
$$= -1.5[\frac{1}{3}x^3 + 2x^2]_{-4}^{0} = -\frac{3}{2}[\frac{64}{3} - 2 * 16] = -32 + 48 = 16.$$

2. Let R be the region bounded by the curves $y = 0.5x(x+1)$ and $y = -3x(x+1)$.

   (a) Sketch

   (b) We have $0.5x(x+1) = -3x(x+1) \iff x(x+1) = 0 \iff x = 0$ or $x = -1$. Moreover the curve $y = 0.5x(x+1)$ lies below the curve $y = -3x(x+1)$ for $-1 < x < 0$. We conclude that the area in question is

$$A = \int_{-1}^{0} -3x(x+4) - 0.5x(x+4)dx = -3.5\int_{-1}^{0} x(x+1)dx$$
$$= -3.5[\frac{1}{3}x^3 + \frac{1}{2}x^2]_{-1}^{0} = -\frac{7}{2}[\frac{1}{3} - \frac{1}{2}] = -\frac{7}{2}[\frac{-1}{6}] = \frac{7}{12}.$$

3. Let R be the region bounded by the curves $y = 0.5x(x-1)$ and $y = -2x(x-1)$.

   (a) Sketch

(b) We have $0.5x(x-1) = -3x(x-1) \iff x(x-1) = 0 \iff x = 0$ or $x = 1$. Moreover the curve $y = 0.5x(x-1)$ lies below the curve $y = -2x(x-1)$ for $0 < x < 1$. We conclude that the area in question is

$$A = \int_0^1 -2x(x-1) - 0.5x(x-1)dx = -2.5\int_0^1 x(x-1)dx$$

$$= -2.5[\frac{1}{3}x^3 - \frac{1}{2}x^2]_0^1 = -\frac{5}{2}[\frac{1}{3} - \frac{1}{2}] = -\frac{5}{2}[\frac{-1}{6}] = \frac{5}{12}.$$

4. Let R be the region bounded by the curves $y = -0.5x(x-2)$ and $y = 2x(x-2)$.

(a) Sketch

(b) We have $-0.5x(x-2) = 2x(x-2) \iff x(x-2) = 0 \iff x = 0$ or $x = 2$. Moreover the curve $y = 2x(x-2)$ lies below the curve $y = -0.5x(x-2)$ for $0 < x < 2$. We conclude that the area in question is

$$A = \int_0^2 -0.5x(x-2) - 2x(x-2)dx = -2.5\int_0^1 x(x-2)dx$$

$$= -2.5[\frac{1}{3}x^3 - x^2]_0^2 = -\frac{5}{2}[\frac{8}{3} - 4] = -\frac{5}{2}[\frac{-4}{3}] = \frac{10}{3}.$$

5. Let R be the region bounded by the curves $y = -2x(x+1)$ and $y = 3x(x+1)$.

(a) Sketch

(b) We have $-2x(x+1) = 3x(x+1) \iff x(x+1) = 0 \iff x = 0$ or $x = -1$. Moreover the curve $y = 3x(x+1)$ lies below the curve $y = -2x(x+1)$ for $-1 < x < 0$. We conclude that the area in question is

$$A = \int_{-1}^0 - -2x(x+1) - 3x(x+1)dx = -5\int_0^1 x(x+1)dx$$

$$= -5[\frac{1}{3}x^3 + \frac{1}{2}x^2]_{-1}^0 = -5[\frac{1}{3} - \frac{1}{2}] = -5[\frac{-1}{6}] = \frac{5}{6}.$$

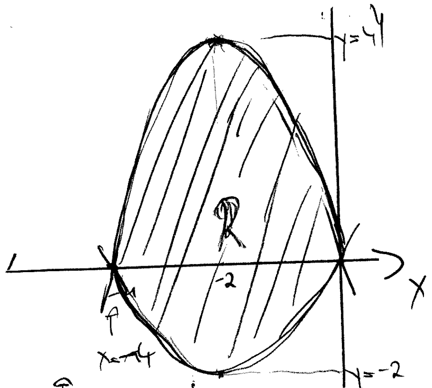
2. Volume calculation (Total=3) Let R be the region bounded by the graph of $y = e^{bx+c}$, the x-axis, the y-axis and the vertical lines $x = u$ and $x = l$. We have

$$V = \pi\int_l^u (e^{bx+c})^2 dx = \pi\int_l^u (e^{2bx+2c})dx = \pi\int_{2bl+2c}^{2bu+2c} \frac{1}{2b}e^z dz$$

1. Area

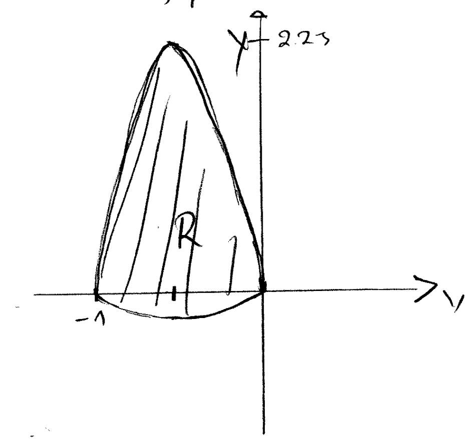
1.

y=0.5x(x+4) , y=-x(x+4)



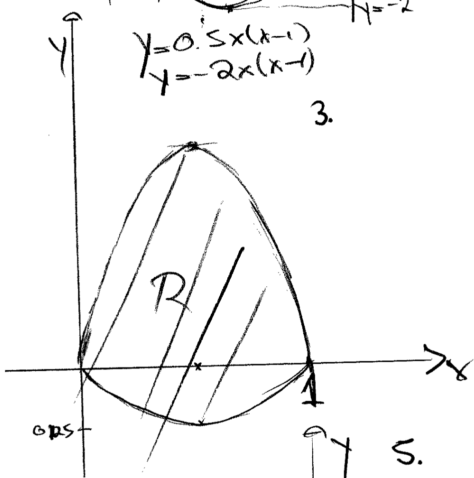
2.

y =0.5x(x+1), y=-3x(x+1)



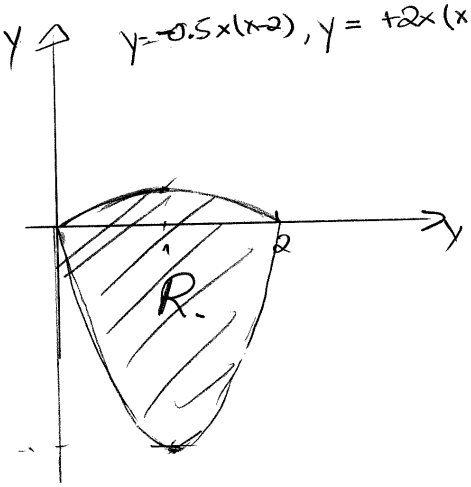
y=0.5x(x-1)
y=-2x(x-1)

3.

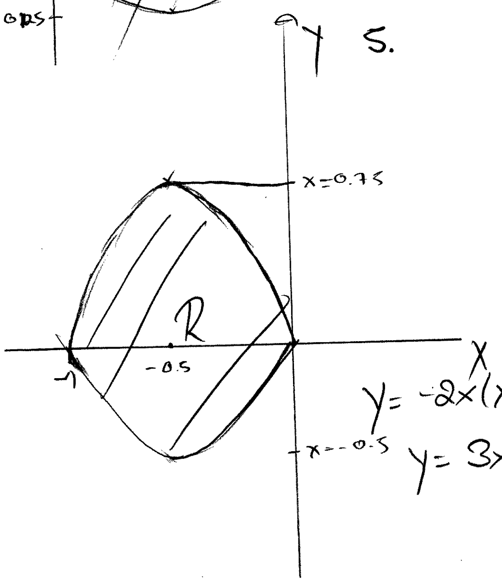


4

y=-0.5x(x-2), y= +2x(x-2)



5.



y= -2x(x+1)

y= 3x(x+1).

where we use the substitution $z = 2bx + 2c$, $dz/dx = 2b$ hence $dx = dz/(2b)$. Therefore the volume of the solid is

$$\pi \int_{2bl+2c}^{2bu+2c} \frac{1}{2b} e^z dz = \frac{\pi}{2b} \left[e^z\right]_{2bl+2c}^{2bu+2c} = \frac{\pi}{2b} \left[e^{2bu+2c} - e^{2bu+2c}\right] = \frac{\pi e^{2c}}{2b} \left[e^{2bu} = e^{2bl}.\right]$$

Marking: 1 point for sketch, 1 point for correctly setting up the integral including correct limits of integration, 0.5 point for correct ant derivative. 0.5 points for correct end result (can be in exact form).

1. Let R be the region bounded by the graph of $y = e^{-x+2}$, the x-axis, the y-axis and the vertical line $x = 4$.

   (a) Sketch

   (b) We have

   $$V = \pi \int_0^4 (e^{-x+2})^2 dx = \pi \int_0^4 (e^{-2x+4}) dx = \pi \int_4^{-4} \frac{1}{-2} e^z dz$$

   where we use the substitution $z = -2x + 4$, $dz/dx = -2$ hence $dx = dz/(-2)$. Therefore the volume of the solid is

   $$\pi \int_4^{-4} \frac{1}{-2} e^z dz = \frac{\pi}{-2} \left[e^z\right]_4^{-4} = \frac{\pi}{-2} \left[e^{-4} - e^4\right] \cong 85.734..$$

2. Let R be the region bounded by the graph of $y = e^{-x+2}$, the x-axis, the y-axis and the vertical line $x = -3$.

   (a) Sketch

   (b) We have

   $$V = \pi \int_{-3}^0 (e^{-x+2})^2 dx = \pi \int_{-3}^0 (e^{-2x+4}) dx = \pi \int_{10}^4 \frac{1}{-2} e^z dz$$

   where we use the substitution $z = 2x + 4$, $dz/dx = 2b$ hence $dx = dz/-2$. Therefore the volume of the solid is

   $$V = \pi \int_{10}^4 \frac{1}{-2} e^z dz = \frac{\pi}{-2} \left[e^z\right]_{10}^4 = \frac{\pi}{-2} \left[e^4 - e^{10}\right] \cong 34513.329.$$

3. Let R be the region bounded by the graph of $y = e^{x+2}$, the x-axis, the y-axis and the vertical line $x = -4$.

(a) Sketch

(b) We have

$$V = \pi \int_{-4}^{0} (e^{x+2})^2 dx = \pi \int_{l}^{u} (e^{2x+4}) dx = \pi \int_{0}^{4} \frac{1}{2} e^z dz$$

where we use the substitution $z = 2x + 4$, $dz/dx = 2$ hence $dx = dz/2$. Therefore the volume of the solid is

$$\pi \int_{0}^{4} \frac{1}{2} e^z dz = \frac{\pi}{2} \left[e^z \right]_{-4}^{4} = \frac{\pi}{2} \left[e^4 - e^{-4} \right] \cong 85.734.$$

4. Let R be the region bounded by the graph of $y = e^{x+1}$, the x-axis, the y-axis and the vertical line $x = -3$.

(a) Sketch

(b) We have

$$V = \pi \int_{-3}^{0} (e^{x+1})^2 dx = \pi \int_{-3}^{0} (e^{2x+2}) dx = \pi \int_{-4}^{2} \frac{1}{2} e^z dz$$

where we use the substitution $z = 2x + 2$, $dz/dx = 2$ hence $dx = dz/(2)$. Therefore the volume of the solid is

$$\pi \int_{-4}^{2} \frac{1}{2} e^z dz = \frac{\pi}{2} \left[e^z \right]_{-4}^{2} = \frac{\pi}{2} \left[e^2 - e^{-4} \right] \cong 11.578.$$

5. Let R be the region bounded by the graph of $y = e^{-x-2}$, the x-axis, the y-axis and the vertical line $x = -3$.

(a) Sketch

(b) We have

$$V = \pi \int_{-3}^{0} (e^{-x-2})^2 dx = \pi \int_{-3}^{0} (e^{-2x-4}) dx = \pi \int_{2}^{-4} \frac{1}{-2} e^z dz$$

where we use the substitution $z = -2x - 4$, $dz/dx = -2$ hence $dx = dz/(-2)$. Therefore the volume of the solid is

$$\pi \int_{2}^{-4} \frac{1}{-2} e^z dz = \frac{\pi}{-2} \left[e^z \right]_{2}^{-4} = \frac{\pi}{-2} \left[e^{-4} - e^2 \right] \cong 11.578$$

3. Differential Equation (Total=5)

1. Solve the following differential equation subject to the initial condition $y(0) = 2$:

$$\frac{dy}{dx} = (y^2 + 4)(2x + 3)$$

To solve the differential equation, we put all the x's on the right-hand side, and all the y's on the left:

$$\frac{dy}{y^2 + 4} = (2x + 3)\, dx.$$

The we integrate:

$$\int \frac{dy}{y^2 + 4} = \int (2x + 3)\, dx.$$

The first integral is calculated as follows:

$$\begin{aligned} \int \frac{dy}{y^2+4} &= \frac{1}{4}\int \frac{dy}{\frac{y^2}{4}+1} = \frac{1}{4}\int \frac{dy}{\left(\frac{y}{2}\right)^2+1} = \frac{1}{4}\int \frac{2du}{u^2+1} = \frac{1}{2}\arctan(u) \\ &= \frac{1}{2}\arctan(y/2), \end{aligned}$$

where we used the substitution $u = y/2$, $du = dy/2$ (so $dy = 2du$). We don't bother with a constant here, since it will be absorbed into the constant in the x integral. The second integral is clearly equal to $x^2 + 3x + C$, with C any constant.

So we now have

$$\frac{1}{2}\arctan(y/2) = x^2 + 3x + C$$

and, solving for y:

$$\begin{aligned} \arctan(y/2) &= 2x^2 + 6x + 2C \\ y/2 &= \tan(2x^2 + 6x + 2C) \\ y &= 2\tan(2x^2 + 6x + 2C). \end{aligned}$$

(Alternatively we could have changed constants, writing C for $2C$, to express the general solution as $y = 2\tan(2x^2 + 6x + C)$, again with C any constant.)

Next we use the initial condition to determine the constant. At $x = 0$, we have $y = 2$, so

$$2 = 2\tan(2 \cdot 0^2 + 6 \cdot 0 + 2C) = 2\tan(2C),$$

and hence $\tan(2C) = 1$, $2C = \pi/4$. So the solution is $\boxed{y = 2\tan(2x^2 + 6x + \pi/4).}$

2. Solve the following differential equation subject to the initial condition $y(0) = 3$:

$$\frac{dy}{dx} = (y^2 + 9)(2x + 4)$$

To solve the differential equation, we put all the x's on the right-hand side, and all the y's on the left:

$$\frac{dy}{y^2 + 9} = (2x + 4)\, dx.$$

The we integrate:

$$\int \frac{dy}{y^2 + 9} = \int (2x + 4)\, dx.$$

The first integral is calculated as follows:

$$\begin{aligned} \int \frac{dy}{y^2 + 9} &= \frac{1}{9} \int \frac{dy}{\frac{y^2}{9} + 1} = \frac{1}{9} \int \frac{dy}{\left(\frac{y}{3}\right)^2 + 1} = \frac{1}{9} \int \frac{3du}{u^2 + 1} = \frac{1}{3} \arctan(u) \\ &= \frac{1}{3} \arctan(y/3), \end{aligned}$$

where we used the substitution $u = y/3$, $du = dy/3$ (so $dy = 3du$). We don't bother with a constant here, since it will be absorbed into the constant in the x integral. The second integral is clearly equal to $x^2 + 4x + C$, with C any constant.

So we now have

$$\frac{1}{3} \arctan(y/3) = x^2 + 4x + C$$

and, solving for y:

$$\begin{aligned} \arctan(y/3) &= 3x^2 + 12x + 3C \\ y/3 &= \tan(3x^2 + 12x + 3C) \\ y &= 3 \tan(3x^2 + 12x + 3C). \end{aligned}$$

(Alternatively we could have changed constants, writing C for $3C$, to express the general solution as $y = 3 \tan(3x^2 + 12x + C)$, again with C any constant.)

Next we use the initial condition to determine the constant. At $x = 0$, we have $y = 3$, so

$$3 = 3 \tan(3 \cdot 0^2 + 12 \cdot 0 + 3C) = 3 \tan(3C),$$

and hence $\tan(3C) = 1$, $3C = \pi/4$. So the solution is $\boxed{y = 3 \tan(3x^2 + 12x + \pi/4).}$

3. Solve the following differential equation subject to the initial condition $y(0) = 4$:

$$\frac{dy}{dx} = (y^2 + 16)(2x + 3)$$

To solve the differential equation, we put all the x's on the right-hand side, and all the y's on the left:

$$\frac{dy}{y^2 + 16} = (2x + 3)\, dx.$$

The we integrate:

$$\int \frac{dy}{y^2 + 16} = \int (2x + 3)\, dx.$$

The first integral is calculated as follows:

$$\begin{aligned}\int \frac{dy}{y^2 + 16} &= \frac{1}{16} \int \frac{dy}{\frac{y^2}{16} + 1} = \frac{1}{16} \int \frac{dy}{\left(\frac{y}{4}\right)^2 + 1} = \frac{1}{16} \int \frac{4du}{u^2 + 1} = \frac{1}{4} \arctan(u) \\ &= \frac{1}{4} \arctan(y/4),\end{aligned}$$

where we used the substitution $u = y/4$, $du = dy/4$ (so $dy = 4du$). We don't bother with a constant here, since it will be absorbed into the constant in the x integral. The second integral is clearly equal to $x^2 + 3x + C$, with C any constant.

So we now have

$$\frac{1}{4} \arctan(y/4) = x^2 + 3x + C$$

and, solving for y:

$$\begin{aligned}\arctan(y/4) &= 4x^2 + 12x + 4C \\ y/4 &= \tan(4x^2 + 12x + 4C) \\ y &= 4\tan(4x^2 + 12x + 4C).\end{aligned}$$

(Alternatively we could have changed constants, writing C for $4C$, to express the general solution as $y = 4\tan(4x^2 + 12x + C)$, again with C any constant.)

Next we use the initial condition to determine the constant. At $x = 0$, we have $y = 4$, so

$$4 = 4\tan(4 \cdot 0^2 + 12 \cdot 0 + 4C) = 4\tan(4C),$$

and hence $\tan(4C) = 1$, $4C = \pi/4$. So the solution is $\boxed{y = 4\tan(4x^2 + 12x + \pi/4).}$

4. Solve the following differential equation subject to the initial condition $y(0) = 5$:

$$\frac{dy}{dx} = (y^2 + 25)(2x + 2)$$

.

To solve the differential equation, we put all the x's on the right-hand side, and all the y's on the left:

$$\frac{dy}{y^2 + 25} = (2x + 2)\, dx.$$

The we integrate:

$$\int \frac{dy}{y^2 + 25} = \int (2x + 2)\, dx.$$

The first integral is calculated as follows:

$$\begin{aligned} \int \frac{dy}{y^2 + 25} &= \frac{1}{25} \int \frac{dy}{\frac{y^2}{25} + 1} = \frac{1}{25} \int \frac{dy}{\left(\frac{y}{5}\right)^2 + 1} = \frac{1}{25} \int \frac{5du}{u^2 + 1} = \frac{1}{5} \arctan(u) \\ &= \frac{1}{5} \arctan(y/5), \end{aligned}$$

where we used the substitution $u = y/5$, $du = dy/5$ (so $dy = 5du$). We don't bother with a constant here, since it will be absorbed into the constant in the x integral. The second integral is clearly equal to $x^2 + 2x + C$, with C any constant.

So we now have

$$\frac{1}{5} \arctan(y/5) = x^2 + 2x + C$$

and, solving for y:

$$\begin{aligned} \arctan(y/5) &= 5x^2 + 10x + 5C \\ y/5 &= \tan(5x^2 + 10x + 5C) \\ y &= 5 \tan(5x^2 + 10x + 5C). \end{aligned}$$

(Alternatively we could have changed constants, writing C for $5C$, to express the general solution as $y = 5 \tan(5x^2 + 10x + C)$, again with C any constant.)

Next we use the initial condition to determine the constant. At $x = 0$, we have $y = 5$, so

$$5 = 5 \tan(5 \cdot 0^2 + 10 \cdot 0 + 5C) = 5 \tan(5C),$$

and hence $\tan(5C) = 1$, $5C = \pi/4$. So the solution is $\boxed{y = 5 \tan(5x^2 + 10x + \pi/4).}$

5. Solve the following differential equation subject to the initial condition $y(0) = 6$:

$$\frac{dy}{dx} = (y^2 + 36)(2x + 1)$$

To solve the differential equation, we put all the x's on the right-hand side, and all the y's on the left:

$$\frac{dy}{y^2 + 36} = (2x + 1)\, dx.$$

The we integrate:

$$\int \frac{dy}{y^2 + 36} = \int (2x + 1)\, dx.$$

The first integral is calculated as follows:

$$\begin{aligned} \int \frac{dy}{y^2 + 36} &= \frac{1}{36} \int \frac{dy}{\frac{y^2}{36} + 1} = \frac{1}{36} \int \frac{dy}{\left(\frac{y}{6}\right)^2 + 1} = \frac{1}{36} \int \frac{6du}{u^2 + 1} = \frac{1}{6} \arctan(u) \\ &= \frac{1}{6} \arctan(y/6), \end{aligned}$$

where we used the substitution $u = y/6$, $du = dy/6$ (so $dy = 6du$). We don't bother with a constant here, since it will be absorbed into the constant in the x integral. The second integral is clearly equal to $x^2 + x + C$, with C any constant.

So we now have

$$\frac{1}{6} \arctan(y/6) = x^2 + x + C$$

and, solving for y:

$$\begin{aligned} \arctan(y/6) &= 6x^2 + 6x + 6C \\ y/6 &= \tan(6x^2 + 6x + 6C) \\ y &= 6 \tan(6x^2 + 6x + 6C). \end{aligned}$$

(Alternatively we could have changed constants, writing C for $6C$, to express the general solution as $y = 6 \tan(6x^2 + 6x + C)$, again with C any constant.)

Next we use the initial condition to determine the constant. At $x = 0$, we have $y = 6$, so

$$6 = 6 \tan(6 \cdot 0^2 + 6 \cdot 0 + 6C) = 6 \tan(6C),$$

and hence $\tan(6C) = 1$, $6C = \pi/4$. So the solution is $\boxed{y = 6 \tan(6x^2 + 6x + \pi/4).}$

4. Average Values (Total=5)

1. What is the average value of the function $f(x) = 2x\sin(3x)$ on the interval $[0, \pi]$?

   The average value is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi - 0}\int_0^\pi 2x\sin(3x)\,dx = \frac{2}{\pi}\int_0^\pi x\sin(3x)\,dx.$$

   We evaluate the integral using integration by parts. We set $u = x$, $dv = \sin(3x)\,dx$, so that $du = dx$ is simpler and $v = -\frac{1}{3}\cos(3x)$ is not more complicated. Then

$$\begin{aligned}\int_0^\pi x\sin(3x)\,dx &= x\left(\frac{1}{3}\cos(3x)\right)\Big|_0^\pi - \int_0^\pi \left(-\frac{1}{3}\cos(3x)\right)\,dx \\ &= x\left(\frac{1}{3}\cos(3x)\right)\Big|_0^\pi + \frac{1}{3}\int_0^\pi \cos(3x)\,dx \\ &= x\left(\frac{1}{3}\cos(3x)\right)\Big|_0^\pi + \frac{1}{9}\sin(3x)\Big|_0^\pi \\ &= \frac{\pi}{3},\end{aligned}$$

   since $\sin(3\pi) = 0$ and both functions evaluate to 0 at 0. Since

$$\frac{2}{\pi}\int_0^\pi x\sin(3x)\,dx = \frac{2}{\pi}\cdot\frac{\pi}{3} = \frac{2}{3},$$

   the average value of $f(x) = 2x\sin(3x)$ on the interval $[0, \pi]$ is $\boxed{\frac{2}{3}.}$

2. What is the average value of the function $f(x) = x\cos(2x)$ on the interval $[0, \pi]$?

   The average value is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi - 0}\int_0^\pi x\cos(2x)\,dx = \frac{1}{\pi}\int_0^\pi x\cos(2x)\,dx.$$

   We evaluate the integral using integration by parts. We set $u = x$, $dv = \cos(2x)\,dx$, so that $du = dx$ is simpler and $v = \frac{1}{2}\sin(2x)$ is not more complicated. Then

$$\begin{aligned}\int_0^\pi x\cos(2x)\,dx &= x\left(\frac{1}{2}\sin(2x)\right)\Big|_0^\pi - \int_0^\pi \left(\frac{1}{2}\sin(2x)\right)\,dx \\ &= x\left(\frac{1}{2}\sin(2x)\right)\Big|_0^\pi - \frac{1}{2}\int_0^\pi \sin(2x)\,dx \\ &= x\left(\frac{1}{2}\sin(2x)\right)\Big|_0^\pi - \frac{1}{4}\cos(2x)\Big|_0^\pi \\ &= -\frac{-1-1}{4} = \frac{1}{2},\end{aligned}$$

since $\sin(2\pi) = \sin(0) = 0$. Since

$$\frac{1}{\pi} \int_0^{\pi} x\cos(2x)\,dx = \frac{1}{\pi} \cdot \frac{1}{2} = \frac{1}{2\pi},$$

the average value of $f(x) = x\cos(2x)$ on the interval $[0,\, \pi]$ is $\boxed{\frac{1}{2\pi}.}$

3. What is the average value of the function $f(x) = 3x\cos x$ on the interval $[0,\, \pi/2]$?

   The average value is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} 3x\cos(x)\,dx = \frac{3}{\pi/2}\int_0^{\pi/2} x\cos(x)\,dx.$$

   We evaluate the integral using integration by parts. We set $u = x$, $dv = \cos(x)\,dx$, so that $du = dx$ is simpler and $v = \sin(x)$ is not more complicated. Then

$$\begin{aligned}\int_0^{\pi/2} x\cos(x)\,dx &= x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi}\sin(x)\,dx \\ &= x\sin(x)\Big|_0^{\pi/2} + \cos(x)\Big|_0^{\pi/2} \\ &= (\pi/2\cdot 1 - 0\cdot 0) + (0-1) = \pi/2 - 1.\end{aligned}$$

   Since

$$\frac{1}{\pi/2}\int_0^{\pi/2} 3x\cos(x)\,dx = \frac{3}{\pi/2}\cdot(\pi/2 - 1) = 3 - \frac{6}{\pi},$$

   the average value of $f(x) = x\cos(3x)$ on the interval $[0,\, \pi/2]$ is $\boxed{3 - \frac{6}{\pi}.}$

4. What is the average value of the function $f(x) = x\cos(3x)$ on the interval $[0,\, \pi]$?

   The average value is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi - 0}\int_0^{\pi} x\cos(3x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos(3x)\,dx.$$

   We evaluate the integral using integration by parts. We set $u = x$, $dv = \cos(3x)\,dx$, so that $du = dx$ is simpler and $v = \frac{1}{3}\sin(3x)$ is not more complicated. Then

$$\begin{aligned}\int_0^{\pi} x\cos(3x)\,dx &= x\left(-\frac{1}{3}\sin(3x)\right)\Big|_0^{\pi} - \int_0^{\pi}\left(\frac{1}{3}\sin(3x)\right)\,dx \\ &= x\left(-\frac{1}{3}\sin(3x)\right)\Big|_0^{\pi} - \frac{1}{3}\int_0^{\pi}\sin(3x)\,dx \\ &= x\left(-\frac{1}{3}\sin(3x)\right)\Big|_0^{\pi} - \frac{1}{9}\cos(3x)\Big|_0^{\pi} \\ &= -\frac{-1-1}{9} = \frac{2}{9},\end{aligned}$$

since $\sin(3\pi) = \sin(0) = 0$. Since

$$\frac{1}{\pi}\int_0^{\pi} x\cos(3x)\,dx = \frac{1}{\pi}\cdot\frac{2}{9} = \frac{2}{9\pi},$$

the average value of $f(x) = x\cos(3x)$ on the interval $[0,\,\pi]$ is $\boxed{\frac{2}{9\pi}.}$

5. What is the average value of the function $f(x) = x\sin x$ on the interval $[0,\,\pi/2]$?

   The average value is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi/2-0}\int_0^{\pi/2} x\sin(x)\,dx = \frac{1}{\pi/2}\int_0^{\pi/2} x\sin(x)\,dx.$$

   We evaluate the integral using integration by parts. We set $u = x$, $dv = \sin(x)\,dx$, so that $du = dx$ is simpler and $v = -\cos(x)$ is not more complicated. Then

$$\begin{aligned}\int_0^{\pi/2} x\sin(x)\,dx &= -x\cos(x)\Big|_0^{\pi/2} - \int_0^{\pi}(-\cos(x))\,dx\\ &= -x\cos(x)\Big|_0^{\pi/2} + \sin(x)\Big|_0^{\pi/2}\\ &= -(\pi/2\cdot 0 - 0\cdot 1) + (1-0) = 1.\end{aligned}$$

   Since

$$\frac{1}{\pi/2}\int_0^{\pi/2} x\sin(x)\,dx = \frac{1}{\pi/2}\cdot(1) = \frac{2}{\pi},$$

   the average value of $f(x) = x\sin(x)$ on the interval $[0,\,\pi/2]$ is $\boxed{\frac{2}{\pi}.}$

5. Definite Integrals (Total=4) Marking: 2 points each: 1 point for integration rules, 1 point for end result. -0.5 for minor mistakes.

1. (a) $\int_0^1 6t(3t^2-1)^5dt = \int_{-1}^2 u^5du$ with $u = 3t^2-1$ and $du/dt = 6t$.

$$\int_0^1 6t(3t^2-1)^5dt = [\frac{1}{6}u^6]_-1^2 = \frac{1}{6}(64-1) = 10.5$$

   (b) $\int_1^e x^3\ln(x)dx = [\ln(x)\frac{1}{4}x^4]_1^e - \int_1^e \frac{1}{x}\frac{1}{4}x^4dx$, by integration by parts ($u = \ln(x), u' = 1/x, v' = x^3, v = x^4/4$). Then

$$[\ln(x)\frac{1}{4}x^4]_1^e - \int_1^e \frac{1}{x}\frac{1}{4}x^4dx = \frac{e^4}{4} - [\frac{1}{4(4)}x^4]_1^e = \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16}.$$

2. (a) $\int_0^{\sqrt{\pi}} x\sin(x^2)dx$. Integration by substitution, $y = x^2, dy/dx = 2x$.

$$\frac{1}{2}\int_0^{\sqrt{\pi}} x\sin(x^2)dx = \frac{1}{2}\int_0^{\pi} \sin(y)dy = \frac{1}{2}[-\cos(y)]_0^{\pi} = \frac{1}{2}(1-(-1)) = 1.$$

(b) $\int_0^1 xe^{2x}dx$. Integration by parts with $u = x, u' = 1, v' = e^{2x}, v = e^{2x}/2$ gives: $\int_0^1 xe^{2x}dx = [\frac{1}{2}xe^{2x}]_0^1 - \int_0^1 \frac{1}{2}e^{2x}dx = [\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}]_0^1 = \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}e^2 + \frac{1}{4}$

3. (a) $\displaystyle\int_1^e \frac{\sqrt{\ln(x)}}{x}dx$.

Substitute: $y = \ln(x), dy/dx = 1/x$, so $\displaystyle\int_1^e \frac{\sqrt{\ln(x)}}{x}dx = \int_0^1 \sqrt{y}dy = [\frac{2}{3}\sqrt{y}^3]_0^1 = \frac{2}{3}$.

(b) $\int_0^1 xe^{-x}dx$. Integration by parts with $u = x, u' = 1, v' = e^{-x}, v = -e^{-x}$ gives $\int_0^1 xe^{-x}dx = [-xe^{-x}]_0^1 + \int_0^1 e^{-x}dx = [-xe^{-x} - e^{-x}]_0^1 = -e^{-1} - e^{-1} + 1 = 1 - 2e^{-1}$.

4. (a) $\int_0^{\pi/2} \sin(2x)e^{\cos(2x)}dx$.
Substitute $y = \cos(2x), dy/dx = -2\sin(2x)$, this gives

$$\int_0^{\pi/2} \sin(2x)e^{\cos(2x)}dx = \frac{-1}{2}\int_1^{-1} e^y dy = 1/2 * [e^y]_{-1}^1 = 1/2(e - e^{-1}).$$

(b) $\int_0^{\pi/2} x\cos(2x)dx$. Integration by parts with $u = x, u' = 1, v' = \cos(2x), v = \sin(2x)/2$ gives

$$\int_0^{\pi/2} x\cos(2x)dx = [\frac{1}{2}sin(2x)x]_0^{\pi/2} - \int_0^{\pi/2} \sin(2x)/2dx$$

$$= [\frac{1}{2}sin(2x)x + \frac{1}{4}cos(2x)]_0^{\pi/2} = \frac{1}{4}\cos(\pi) - \frac{1}{4} = -\frac{1}{2}.$$

5. (a) $\int_0^2 \left(\frac{1}{2}t - 1\right)^{13} dt$. We substitute $u = \frac{1}{2}t - 1, du/dt = 1/2$ and thus : $\int_0^2 \left(\frac{1}{2}t - 1\right)^{13} dt = 2\int_{-1}^0 u^{13}du = \frac{2}{14}[u^{14}]_{-1}^0 = \frac{-2}{14}$.

(b) $\int_0^{\pi/3} x\sin(3x)dx$. Use integration by parts $u = x, u' = 1, v' = \sin(3x), v = -\cos(3x)/3$ and then

$$\int_0^{\pi/3} x\sin(3x)dx = [-x\cos(3x)/3] + \int_0^{\pi/3} \cos(3x)/3dx = [-x\cos(3x)/3 + \sin(3x)/9]_0^{\pi/3} = \pi/3.$$

6. Zombies (Total=4)

Zombies have invaded campus! Initially, there are 5 zombies. They recruit more of the undead to their ghoulish ranks at rate

$$\frac{dz}{dt} = ate^{-bt},$$

where t is the time in days and z are the number of zombies. How many will be infected if the zombies recruited forever?

Solution. Using integration by parts, we have

$$u = at \qquad v' = e^{-bt}$$

$$u' = a \qquad v = -\frac{e^{-bt}}{b}$$

Thus

$$\begin{aligned} z &= -\frac{ate^{-bt}}{b} + \frac{a}{b}\int e^{-bt}dt \\ &= -\frac{ate^{-bt}}{b} - \frac{a}{b^2}e^{-bt} + C \end{aligned}$$

Using the initial condition $z(0) = 5$, we have

$$z(0) = -0 - \frac{a}{b^2}e^0 + C = -\frac{a}{b^2} + C = 5\,.$$

Thus,

$$C = 5 + \frac{a}{b^2}\,.$$

and hence

$$z(t) = -\frac{ate^{-bt}}{b} - \frac{a}{b^2}e^{-bt} + 5 + \frac{a}{b^2}\,.$$

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty}\left[-\frac{ate^{-bt}}{b} - \frac{a}{b^2}e^{-bt} + 5 + \frac{a}{b^2}\right]\,.$$

The last three terms aren't a problem, but the first one might be, because

$$\lim_{t\to\infty} -\frac{ate^{-bt}}{b} = \infty \times 0$$

which we *cannot* evaluate as is. It's an indeterminant form and there's only one way to deal with those: L'Hôpital's rule. Which means we need to have it in a fractional form. Thus

$$\lim_{t\to\infty} -\frac{ate^{-bt}}{b} = \lim_{t\to\infty} -\frac{at}{be^{bt}} = \frac{\infty}{\infty}\,.$$

Hence, we can apply L'Hôpital's rule:

$$\lim_{t\to\infty} -\frac{at}{be^{bt}} = \lim_{t\to\infty} -\frac{a}{b^2e^{bt}} = 0\,.$$

We thus have

$$\begin{aligned}\lim_{t\to\infty} z(t) &= \lim_{t\to\infty}\left[-\frac{ate^{-bt}}{b} - \frac{a}{b^2}e^{-bt} + 5 + \frac{a}{b^2}\right]\\ &= 0 - 0 + 5 + \frac{a}{b^2}\end{aligned}$$

- **Version 1**: $a = 8$ and $b = 0.07$ so the answer is $5 + \frac{8}{0.07^2} = 1637.65$, which means 1637 zombies.
- **Version 2**: $a = 7$ and $b = 0.06$ so the answer is $5 + \frac{7}{0.06^2} = 1949.44$, which means 1949 zombies.
- **Version 3**: $a = 12$ and $b = 0.09$ so the answer is $5 + \frac{12}{0.09^2} = 1481.48$, which means 1481 zombies.

(Note that we always round down because we're talking about individuals.)

7. Partial Fraction/Improper Integrals (Total=6) Marking: (a) 1 point, you must mention that the function is discontinuous at one of the end points, (b) 0.5 for factoring the denominator, 0.5 for setting up partial fractions as $\frac{A}{x-a} + \frac{B}{x-b}$, 0.5 points for finding A, 0.5 points for findinb B, 1 point for correct indefinite integral $A\ln|x-a| + B\ln|x-b| + C$. (c) 1 point for correct answer, 1 point for justification (you must show that you know that improper integrals are definite via limits to earn full points).

1. Given the integral

$$\int_0^2 \frac{x-4}{x^2-5x+6}dx \tag{1}$$

(a) Explain carefully why the integral (1) is an improper integral.

The integral is improper because the function $f(x) = \frac{x-4}{x^2-5x+6}$ is not continuous at

$x = 2$ which is the right limit of integration.
(b) Using partial fractions, find the indefinite integral

$$\int \frac{x-4}{x^2-5x+6}dx.$$

The denominator factors as $(x-2)(x-3)$. Hence we must find A and B such that

$$\frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{x^2-5x+6} = \frac{x-4}{x^2-5x+6}.$$

The equations for the numerator are

$$A + B = 1$$

and

$$-3A - 2B = -4$$

From the first equation, $B = 1 - A$ and if we plug this information in the second equation, we get so $-A = 1$, hence $A = -1, B = 2$. The indefinite integral is

$$\int \frac{x-4}{x^2-5x+6}dx = \int \frac{-1}{x-2} + \frac{2}{x-3}dx = -\ln|x-2| + 2\ln|x-3| + C.$$

(c) Is the integral (1) convergent or divergent? Justify your answer. If it is convergent evaluate the integral. The integral is convergent if and only if the limit

$$\lim_{t\to 2^-} \int_0^t \frac{x-4}{x^2-5x+6}dx = \lim_{t\to 2^-} [-\ln|x-2| + 2\ln|x-3|]_0^t$$

exists. We evaluate as far as we can:

$$\lim_{t\to 2^-} [-\ln|x-2| + 2\ln|x-3|]_0^t = -\lim_{t\to 2^-} [-\ln|x-2|] + 2\ln|1| + \ln|2| - 2\ln|1|].$$

But this limit does not exist, hence the integral is not convergent.

2. Given the integral

$$\int_3^4 \frac{2x-3}{x^2-5x+6}dx \qquad (2)$$

(a) The integral is improper because the function $f(x) = \frac{2x-3}{x^2-5x+6}$ is not continuous at $x = 3$ which is the left limit of integration.
(b) The denominator factors as $(x-2)(x-3)$. Hence we must find A and B such that

$$\frac{A}{x-3} + \frac{B}{x-2} = \frac{A(x-2)+B(x-3)}{x^2-5x+6} = \frac{2x-3}{x^2-5x+6}.$$

The equations for the numerator are

$$A + B = 2$$

and

$$-2A - 3B = -3$$

From the first equation, $B = 2 - A$ and if we plug this information in the second equation, we get so $-A = 1$, hence $A = -1, B = 3$. The indefinite integral is

$$\int \frac{2x-3}{x^2-5x+6}dx = \int \frac{-1}{x-3} + \frac{3}{x-2}dx = -\ln|x-3| + 3\ln|x-2| + C.$$

(c) Is the integral (1) convergent or divergent? Justify your answer. If it is convergent evaluate the integral. The integral is convergent if and only if the limit

$$\lim_{t\to 3^+} \int_t^4 \frac{x-4}{x^2-5x+6}dx = \lim_{t\to 3^+} [-\ln|x-3| + 3\ln|x-2|]_t^4$$

exists. We evaluate as far as we can:

$$\lim_{t\to 3^+} [-\ln|x-3| + 3\ln|x-2|]_t^4 = [-\ln|1| + 3\ln|2|] - \lim_{t\to 3^+} [-\ln|t-3| + 3\ln|1|].$$

But this limit does not exist, hence the integral is not convergent.

3. Given the integral

$$\int_0^2 \frac{-2x+3}{x^2-x-2}dx \tag{3}$$

(a) The integral is improper because the function $f(x) = \frac{-2x+3}{x^2-x-2}$ is not continuous at $x = 2$ which is the right limit of integration.
(b) The denominator factors as $(x-2)(x+1)$. Hence we must find A and B such that

$$\frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2) + B(x+1)}{x^2-x-2} = \frac{-2x+3}{x^2-x-2}.$$

The equations for the numerator are

$$A + B = -2$$

and

$$-2A + B = 3$$

From the first equation, $B = -2 - A$ and if we plug this information in the second equation, we get so $A = -1/3$, hence $A = -1/3, B = -5/3$. The indefinite integral is

$$\int \frac{-2x+3}{x^2-x-2}dx = \int \frac{-1}{3(x+1)} + \frac{-5}{3(x-2)}dx = -1/3\ln|x+1| - 5/3\ln|x-2| + C.$$

(c) The integral is convergent if and only if the limit

$$\lim_{t\to 2^-}[-1/3\ln|x+1| - 5/3\ln|x-2|]_0^t$$

exists. We evaluate as far as we can:

$$\lim_{t\to 2^-}[-1/3\ln|3| + 5/3\ln|1| + 1/3\ln|1| - 5/3\ln|t-2|].$$

But this limit does not exist, hence the integral is not convergent.

4. Given the integral

$$\int_1^2 \frac{3x-5}{x^2-5x+6}dx \tag{4}$$

(a) The integral is improper because the function $f(x) = \frac{3x-5}{x^2-x-2}$ is not continuous at $x = 2$ which is the right limit of integration.
(b) The denominator factors as $(x-2)(x-3)$. Hence we must find A and B such that

$$\frac{A}{x-3} + \frac{B}{x-2} = \frac{3x-5}{x^2-5x+6}.$$

The equations for the numerator are

$$A + B = 3$$

and

$$-2A - 3B = -5$$

From the first equation, $B = 3 - A$ and if we plug this information in the second equation, we get so $A = 4$, hence $A = 4, B = -1$. The indefinite integral is

$$\int \frac{3x-5}{x^2-5x+6}dx = \int \frac{4}{(x-3)} + \frac{-1}{(x-3)}dx = 4\ln|x-3| - \ln|x-2| + C.$$

(c) The integral is convergent if and only if the limit

$$\lim_{t\to(2)^-}[-4\ln|x-3| - \ln|x-3|]_1^t$$

exists. We evaluate as far as we can and get

$$\lim_{t\to 2^-}[-\ln|t-2| - 4\ln|2|]$$

But this limit does not exist, hence the integral is not convergent.

5. Given the integral

$$\int_{-1}^{0} \frac{-x-3}{x^2-x-2} dx \tag{5}$$

(a) The integral is improper because the function $f(x) = \frac{-x-3}{x^2-x-2}$ is not continuous at $x = -1$ which is the right limit of integration.
(b) The denominator factors as $(x-2)(x+1)$. Hence we must find A and B such that

$$\frac{A}{x+1} + \frac{B}{x-2} = \frac{-x-3}{x^2-x-2}.$$

The equations for the numerator are

$$A + B = -1$$

and

$$-2A + B = -3$$

From the first equation, $B = -1 - A$ and if we plug this information in the second equation, we get so $A = 2/3$, hence $A = 2/3, B = -5/3$. The indefinite integral is

$$\int \frac{-x-3}{x^2-x-2} dx = \int \frac{2}{3(x+1)} + \frac{-5}{3(x-2)} dx = \frac{2}{3} \ln|x+1| - \frac{-5}{3} \ln|x-2| + C.$$

(c) The integral is convergent if and only if the limit

$$\lim_{t \to (-1)^+} [\frac{2}{3} \ln|x+1| - \frac{-5}{3} \ln|x-2|]_{-1}^{0}$$

exists. We evaluate as far as we can and get

$$\lim_{t \to -1^+} [2/3 \ln|t+1| + 5/3 \ln|2| - 5/3 \ln|3|]$$

But this limit does not exist, hence the integral is not convergent.