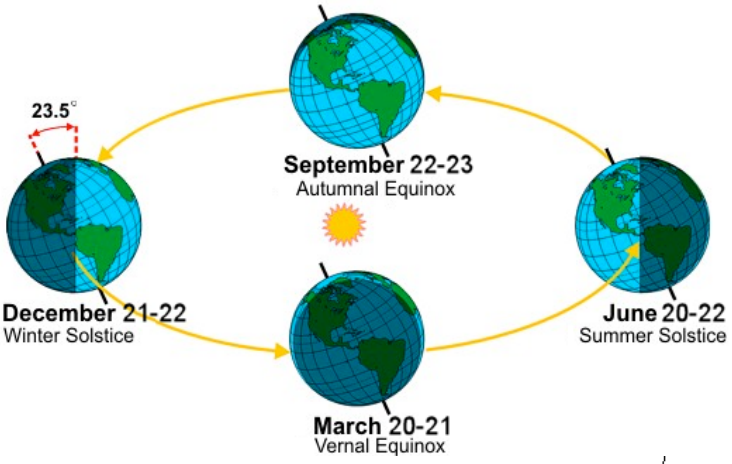
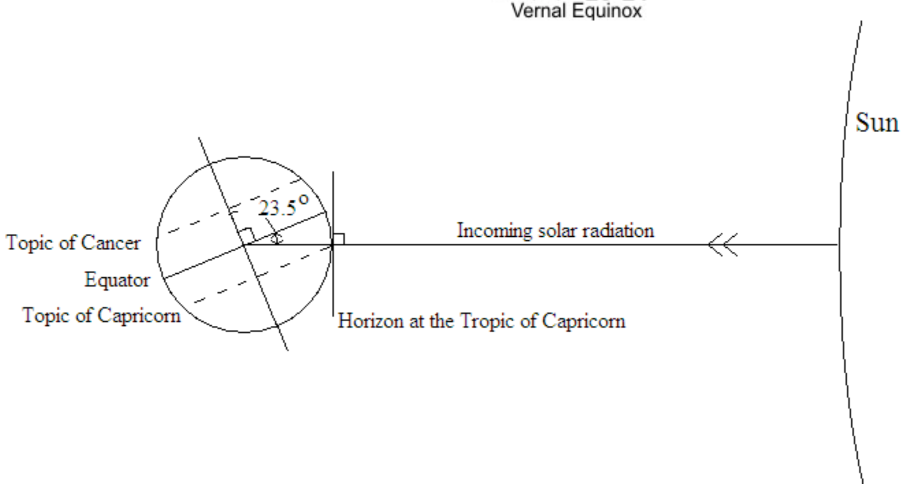
CIV 300: PPS 1 Solutions

Winter 2012

1. Toronto is found at approximately 44.5 degrees north latitude. (i) How far above the horizon does the sun achieve at solar noon on the shortest day of the year in the north hemisphere (21 Dec.)? Also what is this angle: (ii) on the equator? (iii) on the Tropic of Capricorn? (iv) At 60 degrees north latitude? (v) At 45 degrees south?

Recall two things: first that there is always some part of the Earth directly facing the sun, and that the Tropics of Capricorn (23.5ºS) is the most southerly latitude at which the sun can appear directly overhead (90° above the horizon) at solar noon on December 21. This is when the southern hemisphere is tilted towards the sun at its maximum extent.



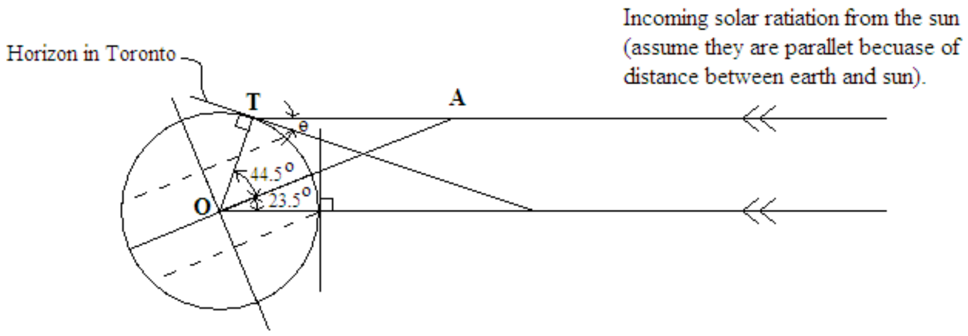


Thus, if you have good intuition you can argue that every degree that you are north or south of the latitude where the sun is directly overhead will shift the sun a degree less than 90. This allows direct calculation of the required angles. Thus, at 60ºN, you are (60+23.5 = 83.5 degrees from this, and the sun is only 6.5° above the horizon at solar noon.

Mathematically, if you let northern latitudes be positive, and southern negative, then the angle is 90 – abs(latitude where the sun is directly overhead – current latitude).

Or you can argue more geometrically as shown on the next page:

1. i)



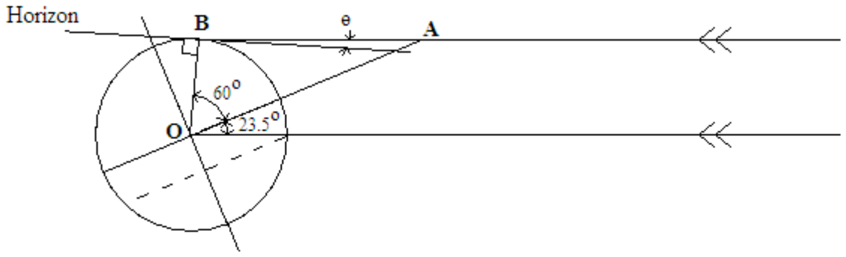
Using geometry

1. Extend equator to the incoming solar radiation to form point A.
2. Label Toronto T and the center of the earth O.
3. ∠OAT = 23.5° (parallel lines, Z)
4. ∠OTA = 180° – 44.5° – 23.5° = 112° (Triangle)
5. θ = 112° – 90 = 22°

1. ii) Using geometry or direct reasoning, θ = 66.5°

1. iii) As reasoned above, this angle clearly 90° (sun directly overhead at noon)

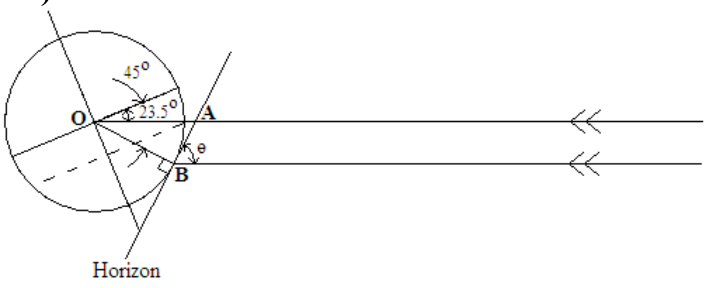
1. iv)



Using geometry

1. Extend equator to the incoming solar radiation to form point A.
2. ∠OAB = 23.5° (parallel lines, Z)
3. ∠OBA = 180° – 60° – 23.5° = 96.5° (Triangle)
4. θ =∠OBA – 90° = 6.5°

1. v)



Using geometry

1. ∠AOB = 45° – 23.5° = 21.5°
2. ∠OAB = 180° – 90° – 21.5° = 68.5° (Triangle)
3. θ =∠OAB = 68.5° (parallel lines, Z)

2. (i) Calculate the average orbital speed (in m/s and km/h) of the Earth around the sun. (ii) What is its average acceleration toward the center, and what produced the required force to achieve this acceleration? (iii) Newton's law of gravitation states that the gravitational attraction between two bodies is proportional to the product of their masses and inversely proportional to the square of the distance separating their centers of mass. The proportionality constant is usually taken as 6.67×10^{-11} N m^2/kg^2. Use this law and the fact that an object on the Earth's surface experiences an acceleration of 9.81 m/s^2 to deduce the mass of the Earth. Use the acceleration of the Earth in its orbit to deduce the mass of the sun. Compare your answers with generally accepted values of these two astronomical constants.

Assume earth's path around the sun is circular (which is indeed an excellent approximation).

2. i) Determine the length of the earth's orbit

$$\begin{aligned} L &= 2\pi R_{oe} \\ &= 2\pi\left(150\times10^{9}\,m\right) \\ &= 9.42\times10^{11}\,m \end{aligned}$$

Determine the duration of the orbit (365 days): $t = 365 * 86400 = 31{,}536{,}000\text{s}$

The average orbital speed

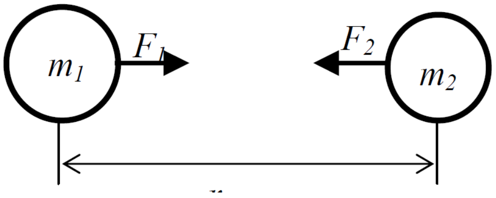
$$\begin{aligned} v &= \frac{L}{t} \\ &= \frac{9.42\times10^{11}\,m}{31{,}536{,}000 s} \\ &= 3.0\times10^{4}\,\tfrac{m}{s} = 1.1\times10^{5}\,\tfrac{km}{hr} \end{aligned}$$

2. ii) Determine the average acceleration

$$\begin{aligned} a &= \frac{v^2}{R_{oe}} \\ &= \frac{\left(3.0\times10^{4}\,\tfrac{m}{s}\right)^2}{150\times10^{9}\,m} \\ &= 6.0\times10^{-3}\,\tfrac{m}{s^2} \end{aligned}$$

This force is achieved by the gravitation attraction between the earth and the sun.

2. iii) Newton's law of gravitation is illustrated below



$$F = G\frac{m_1 m_2}{r^2} \qquad F = F_1 = F_2$$

To determine the mass of the earth, let m_1 be the mass of the falling object and m_2 be the mass of the earth. From Newton's second law the net force on the falling object is $F = ma$

Thus,

$$F = m_1 g$$

So,

$$\begin{aligned} m_1 g &= G\frac{m_1 m_2}{R_e^2} \\ m_2 &= \frac{gR_e^2}{G} \\ &= \frac{\left(9.81\,\frac{m}{s^2}\right)\left(6.35\times10^{6}\,m\right)^2}{6.67\times10^{-11}\,\frac{Nm^2}{kg^2}} \\ &= 5.93\times10^{24}\,kg \end{aligned}$$

Generally accepted value is very close to this: $5.98\times10^{24}\,kg$

To determine the mass of the sun let m_1 be the mass of sun and m_2 be the mass of the earth; r is the radius of earth's orbit around the sun; $F = m_2 a$ where $a = 6.0\times10^{-3}\,\frac{m}{s^2}$ (from 2. ii).

$$\begin{aligned} m_2 a &= G\frac{m_1 m_2}{R_{oe}^2} \\ m_1 &= \frac{aR_{oe}^2}{G} \\ &= \frac{\left(6.0\times10^{-3}\,\frac{m}{s^2}\right)\left(150\times10^{9}\,m\right)^2}{6.67\times10^{-11}\,\frac{Nm^2}{kg^2}} \\ &= 2.02\times10^{30}\,kg \end{aligned}$$

Generally accepted value is $1.99\times10^{30}\,kg$

3. (i) What is the rotational velocity (in m/s and km/h) on the equator and at 44 degrees north latitude? (ii) What is its average acceleration toward the center on the equator, and what produced the required force to achieve this acceleration?

At the equator earth's diameter is $12.7\times10^{6}\,m$

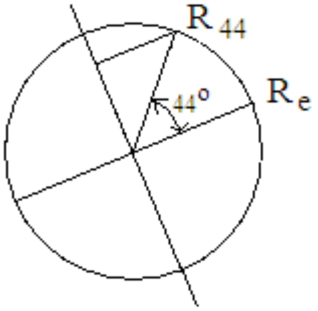
3. i) Determine earth's circumference

$$\begin{aligned} L &= \pi D \\ &= \pi\left(12.7\times10^{6}m\right) \\ &= 4.0\times10^{7}m \end{aligned}$$

Determine duration of rotation of 1 day $t = 86{,}400s$; this gives the rotational velocity

$$\begin{aligned} v &= \frac{L}{t} \\ &= \frac{4.0\times10^{7}m}{86{,}400s} \\ &= 463.0\tfrac{m}{s} = 1666.8\frac{km}{hr} \end{aligned}$$

Looking at 44 degrees north



$$\begin{aligned} R_{44} &= R_e\cos 44^{o} \\ &= \left(6.35\times10^{6}m\right)\cos 44^{o} \\ &= 4.57\times10^{6}m \end{aligned}$$

Rotational velocity

$$\begin{aligned} v &= \frac{L}{t} \\ &= \frac{2\pi R_{44}}{86{,}400s} \\ &= 332.3\tfrac{m}{s} = 1196.3\frac{km}{hr} \end{aligned}$$

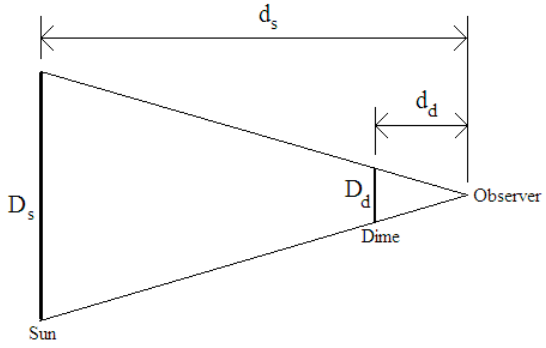
3. ii)

$$\begin{aligned} a &= \frac{v^{2}}{R_e} \\ &= \frac{\left(463.0\tfrac{m}{s}\right)^{2}}{6.35\times10^{6}m} \\ &= 0.034\tfrac{m}{s^{2}} \end{aligned}$$

The earth's rotation causes the surface to move fastest at the equator and not at all at the poles. This causes an apparent force, the so-called Coriolis force, since gravity and the acceleration are no longer collinear.

4. (i) At what distance from you should a dime (a 10 cent piece, with an assumed or measured diameter of 1.8 cm) be placed such that it exactly covers the Sun? How far to cover the Moon? What does this imply about the nature of a total eclipse? (ii) If the Earth were modeled as a large globe of 1 m in diameter, how think would a proportional representation be of the average ocean depth (4 km) and the average troposphere thickness (12 km)?

4. i)



$$D_s = 1.39\times10^9 m$$
$$d_s = 150\times10^9 m$$

$$D_d = 0.018m$$

By similar triangles

$$\frac{D_s}{d_s} = \frac{D_d}{d_d}$$
$$\frac{1.39\times10^9 m}{150\times10^9 m} = \frac{0.018m}{d_d}$$
$$D_d = 1.9\text{m}$$

Similarly, for the moon

$$\frac{D_m}{d_m} = \frac{D_d}{d_d}$$
$$\frac{3.48\times10^6 m}{4.0\times10^8 m} = \frac{0.018m}{d_d}$$
$$d_d = 2.1\text{m}$$

Since the distances are almost equal, when a total eclipse occurs the moon just blocks the sun, but leaves only the outer most part so we can see solar flares happening.

4. ii)

The diameter of the Earth is 1.27×10^7 m. So using a ratio the model heights can be determined.

The average depth of the ocean

$$\frac{1m}{1.27\times10^{7}m}=\frac{h_{mo}}{4000m}$$
$$h_{mo}=3.15\times10^{-4}m$$
$$0.315\text{ mm}$$

Which is about the thickness of three average pieces of paper. The average troposphere thickness is obviously three times this value, or about one mm. More formally:

$$\frac{1m}{1.27\times10^{7}m}=\frac{h_{mo}}{12000m}$$
$$h_{mo}=9.45\times10^{-4}m$$
$$0.945\text{ mm}$$

5. Assuming a solar insolation of exactly 1370 W/m^2, (i) how much total energy does the Earth receive from the sun? (ii) Approximately what fraction of the Sun's total energy output does the Earth receive? Why is it clear that the Earth radiates almost the same amount of energy as it receives? (iii) Estimate the variation of solar intensity, as the average Earth distance from the sun, between the near side of the Earth and its far side from the sun.

5. i) Determine the projected area of the earth

$$\begin{aligned}SA_p &= \pi R_e^2\\ &= \pi\left(6.35\times10^{6}m\right)^2\\ &= 1.3\times10^{14}m^2\end{aligned}$$

The energy the earth receives from the sun

$$\begin{aligned}E_e &= (SI)(SA_p)\\ &= \left(1370\tfrac{w}{m^2}\right)\left(1.3\times10^{14}m^2\right)\\ &= 1.7\times10^{17}W\end{aligned}$$

This is a good number for basic reference and worth remembering: the Earth receives a total of about 170 PW from the sun.

5. ii) The total energy passing through a spherical shell at earth's orbit is

$$E_{oe}=(SI)\left(4\pi R_{oe}^2\right)$$

Giving a fraction of

$$\begin{aligned}\frac{E_e}{E_{oe}} &= \frac{(SI)(\pi R_e^2)}{(SI)(4\pi R_{oe}^2)}\\ &= \frac{R_e^2}{4R_{oe}^2}\\ &= \frac{(6.35\times 10^6 m)^2}{4(150\times 10^9 m)^2}\\ &= 4.5\times 10^{-10}\end{aligned}$$

The Earth radiates almost the same amount of energy as it receives because the temperate of the Earth remains about constant when averaged over its entire surface. Of course, there are important variations of great significance, leading to the possibility of global warming, or at other times the initiation of ice ages, or other departures from "expected" or "normal" conditions.

5. iii) According to the inverse-square law, solar intensity at each of these points (Far side and Near side) is inversely proportional to the square of their distance from the sun. Because the surface area of a sphere increases with the square of the radius.

R_{NEO} = Distance from Near side of the Earth to the Sun= $R_{AEO}- R_e$
R_{FEO} = Distance from Far side of the Earth to the Sun= $R_{AEO}+ R_e$
R_e = Radius of the Earth
S_{Near} = Solar intensity at Near side of the Earth
S_{Far} = Solar intensity at Far side of the Earth

$$(S_{Near})(4\pi R_{NEO}^2) = (SI)(4\pi R_{AEO}^2) \Rightarrow S_{Near} = \frac{SI(R_{AEO}^2)}{R_{NEO}^2}$$

$$(S_{Far})(4\pi R_{FEO}^2) = (SI)(4\pi R_{AEO}^2) \Rightarrow S_{Far} = \frac{SI(R_{AEO}^2)}{R_{FEO}^2}$$

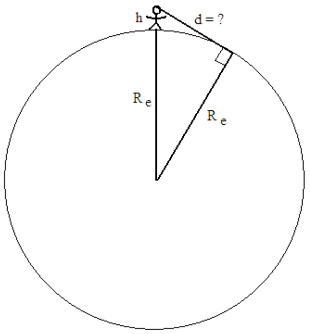
$$S_{Near} - S_{Far} = SI\left[\frac{R_{AEO}^2}{(R_{AEO}-R_e)^2} - \frac{R_{AEO}^2}{(R_{AEO}+R_e)^2}\right] = SI\left[\frac{1}{\left(1-\frac{R_e}{R_{AEO}}\right)^2} - \frac{1}{\left(1+\frac{R_e}{R_{AEO}}\right)^2}\right]$$

R_e = $6.35\times 10^6 m$
R_{AEO} = $150\times 10^9 m$
SI = 1370 W/m^2

$$S_{Near} - S_{Far} = 1370\left[\frac{1}{\left(1-\frac{6.35\times 10^6}{150\times 10^9}\right)^2} - \frac{1}{\left(1+\frac{6.35\times 10^6}{150\times 10^9}\right)^2}\right] = 1370(1.694\times 10^{-4}) = 0.23$$

So the different in intensity between the two sides is about a quarter of a W/m^2. For most purposes, such a variation is trivial. However, by contrast, the variation over the year, as the Earth comes nearer and then moves further from the sun, is slightly more significant, but still a minor issue for most purposes (and this difference it is partly compensated for by the difference in the time the Earth spends in these positions.

6. Calculate the distance to the perceived horizon as a function of an observer's height h. Assume a flat landscape such as the Ocean, a large lake or a flat plateau of prairie.



h = observer's height

Using Pythagorean theorem: $a^2 + b^2 = c^2$

$$
\begin{aligned}
d^2 + R_e^2 &= (h + R_e)^2 \\
d^2 + R_e^2 &= h^2 + 2hR_e + R_e^2 \\
d^2 &= h^2 + 2hR_e \\
d &= \sqrt{h^2 + 2hR_e}
\end{aligned}
$$

For $h = 2$ m $\quad d = 5$ km

For $h = 400$ m (observation deck of CN tower) $\quad d = 71.3$ km

Helpful Simplification: Note in the expression for d that the first h^2 term under the square root sign is negligible for typical values of d (i.e., h << R). This gives a very handy approximation: $d = \sqrt{2hR_e} = \sqrt{hD_e} = 3.6\sqrt{h}$ if h is in m and the distance d is in km.

This hand formula is a easy thing to "carry around with you" to be able to estimate, and visualize, earth horizon distances in a variety of "near Earth" contexts.

It is a helpful way to remind yourself of the curvature of the Earth, something that is surprising easy to disregard. There are endless and important implications of the Earth's shape and motion to terrestrial energy systems.

7. **You are considering the installation of a solar PV system in the in Iqaluit, NT (latitude: 63 North. Real cost of diesel fired electrical generation: over 60 cents per kWh). i) Consider the items impacting the location of the modules and try and list the key inputs to that decision. Ii) Considering that, now think of the same installation in Toronto. What rationale is there for and against installing them flush to a flat roof?**

i) Installation is a matter of practicality, cost and energy generation. Typically a solar system biased for summer production (which this would be given that in winter there is very little sun due to proximity to the Arctic circle – though as discussed in the lecture it can be an option). The rule of thumb for that is to install at latitude minus 15 degrees, resulting in an ideal angle of 48 degrees.

However that ignores practicalities of snow, snow accumulation and snow sticking to the sloped glazed surface, high wind loads generated by a pitched install (we are in a very windy part of the world in the arctic by the coast), cost of installing racking plus concerns over additional snow drifting over-stressing the building's structure. Generally this means comparing the costs of racking and necessary structural reinforcement with the energy you get at the ideal level.

Mounting vertically may be the simplest and easiest though not optimum for solar energy.

Shading. Row on row shading is also an issue for concern: low sun angles at certain times of the year result in a question of how much shade to allow. Typically you calculate the gap required to not have shading at noon on Dec 21st, which in this case would be based on a sun angle of 3 degrees above horizontal, resulting in a close to infinite shade gap. Therefore a compromise is required.

Interesting fact: a solar PV energy model (PVSol) indicates and optimum install angle of 51 degrees (best total insolation).

ii) Flush mount

Positives: low racking cost, no additional wind loading on structure, no shade gaps required meaning space for more modules on a roof

Negatives: snow burial, lack of service access if close together, dirt accumulation/less ability for rain to wash clean.

NOTE this is a question to get your minds thinking about practicalities, in the examination we would only be interested in the physical calculations of the sun in the sky as covered in the lectures and in Q1 here.