

Department of Mathematics, University of Toronto  
**MAT224H1F - Linear Algebra II**  
Fall 2013

**Problem Set 2**

- Due at **12:00 noon** on Wednesday, October 9, in the drop boxes in SS1071.
- Be sure to clearly write your name, student number, your tutorial group and the name of your TA on the top right-hand corner of your assignment, and *staple* the pages together. Remember that the clarity of your solutions matters - your assignment should be a *final* draft, not a *first* draft.
- You are welcome to work in groups but problem sets must be written up independently - any suspicion of copying/plagiarism will be dealt with accordingly. You are welcome to discuss the problem set questions in tutorial, or with your instructor. You may also use Piazza to discuss problem sets but you are not permitted to ask for or post complete solutions to problem set questions.

1. Let  $T : \mathbb{Z}_5^4 \rightarrow M_{2 \times 2}(\mathbb{Z}_5)$  be defined by

$$T(a, b, c, d) = \begin{bmatrix} a - b + 2d & b - d \\ a + d & a + b + c + d \end{bmatrix}.$$

- (a) Is  $T$  injective? If not, find a basis for  $\ker(T)$ .
- (b) Is  $T$  surjective? If not, find a basis for  $\text{im}(T)$ .

2. Let  $T : \mathbb{C}^3 \rightarrow P_1(\mathbb{C})$  be defined by

$$T(a, b, c) = (a + b)x + (b - c).$$

Let  $S : P_1(\mathbb{C}) \rightarrow \mathbb{C}^3$  be defined by

$$S(p(x)) = (p(0), p(-1), p(1)).$$

Let  $\alpha = \{(1, 0, 1), (0, 1, 1), (1, 1, 1)\}$  be a basis for  $\mathbb{C}^3$ ,  $\beta = \{1, 1 - x\}$  be a basis for  $P_1(\mathbb{C})$ , and  $\gamma = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  be a basis for  $\mathbb{C}^3$ .

- (a) Find  $[T]_\alpha^\beta$  and  $[S]_\gamma^\beta$ .
  - (b) Find  $[ST]_\alpha^\gamma$ .
  - (c) Use  $[ST]_\alpha^\gamma$  to find a basis for  $\ker(ST)$  and a basis for  $\text{im}(ST)$ .
3. Let  $\alpha = \{(1, 1, -1), (3, 2, -1), v_3\}$  and  $\beta = \{(0, 0, 1), w_2, w_3\}$  be bases for  $\mathbb{R}^3$  such that the transition matrix from  $\alpha$  to  $\beta$  is

$$\begin{bmatrix} 0 & 1 & -2 \\ 3 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix}.$$

Find  $v_3, w_2, w_3$ .

4. Let  $V$  be the subspace of  $P_3(\mathbb{C})$  spanned by

$$\{ix + 1, x^3 + 2x, 2ix^2 + ix - 1, x^3 - 2x^2\}.$$

Find  $d = \dim(V)$ , and find an isomorphism  $T : V \rightarrow \mathbb{C}^d$ .

5. Let  $\alpha = \{1, 1 + x, 1 + x^2\}$  and  $\beta = \{1, 1 + x, 1 + x + x^2\}$  be bases for  $P_2(\mathbb{C})$ . Let  $T : P_2(\mathbb{C}) \rightarrow P_2(\mathbb{C})$  be the linear function such that

$$[T]_{\alpha}^{\beta} = \begin{bmatrix} i & -i & i \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

- (a) Find a formula for  $T(p(x))$  for every  $p(x) \in P_2(\mathbb{C})$ .  
 (b) Find a formula for  $T^{-1}(p(x))$  for every  $p(x) \in P_2(\mathbb{C})$ .
6. Let  $V, W, X$  be vector spaces over a field  $F$ , and let  $T : V \rightarrow W$  and  $S : W \rightarrow X$  be linear. For each of the following statements, decide if the statement is true or false. If the statement is true, give a proof. If the statement is false, give a counterexample.
- (a) If  $T$  and  $S$  are injective, then  $ST$  is injective.  
 (b) If  $T$  and  $S$  are surjective, then  $ST$  is surjective.  
 (c) If  $ST$  is injective then both  $S$  and  $T$  are injective.  
 (d) If  $ST$  is surjective then both  $S$  and  $T$  are surjective.
7. Let  $V$  and  $W$  be finite-dimensional vector spaces over a field  $F$ .
- (a) Let  $T : V \rightarrow W$  be an injective linear function. Prove that  $V$  is isomorphic to  $\text{im}(T)$ .  
 (b) Suppose that  $1 \leq \dim(V) \leq \dim(W)$ . Show that there is a subspace  $U \subseteq W$  such that  $V \cong U$ .
8. Let  $V$  be a finite-dimensional vector space over a field  $F$ , and let  $W_1$  and  $W_2$  be subspaces of  $V$ . Prove that  $V = W_1 \oplus W_2$  if and only if whenever  $\alpha_1$  is a basis for  $W_1$  and  $\alpha_2$  is a basis for  $W_2$ , then  $\alpha_1 \cup \alpha_2$  is a basis for  $V$ .
9. Let  $V$  be a finite-dimensional vector space over a field  $F$ . Let  $T : V \rightarrow V$  be a linear function such that  $T^2 = T$ , i.e., for all  $v \in V$ , we have  $T(T(v)) = T(v)$ . Prove that  $V = \ker(T) \oplus \text{im}(T)$ . *Hint:* Any  $v \in V$  can be written as  $v = v - T(v) + T(v)$ .

**Suggested Extra Problems (not to be handed in):**

- Textbook, Section 2.4 **1, 2, 5, 6**
- Textbook, Section 2.5 **8, 10, 12**
- Textbook, Section 2.6 **1, 2, 6, 7, 8, 17**
- Textbook, Section 2.7 **1, 2, 3, 4, 9**
- Textbook, Chapter 2 Supplementary Exercises, **1, 2, 3, 4, 5**