2141. F08 . Mdt soln.

1. Let $\mathcal{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$ be the vector space of real-valued functions on $\mathbf{R}$. Define $f, g, h \in \mathcal{F}(\mathbf{R})$ by

$$f(x) = \sin x, \quad g(x) = \cos x, \quad h(x) = \sin(x + \frac{\pi}{6}), \quad \forall x \in \mathbf{R}.$$

2 a) Prove that $\{f, g\}$ is linearly independent.

2 b) Prove that $\{f, g, h\}$ is linearly dependent.

3 c) Find the dimension of the subspace span$\{f, g, h\}$.

$\frac{1}{2}$ knowing what to do

a) Suppose $af + bg = 0$, for $a, b \in \mathbb{R}$. That is, $af(x) + bg(x) = 0$, $\forall x \in \mathbb{R}$.

In particular, at $x = 0$, $a.0 + b.1 = 0$, and (1) Sys implying $a = b = 0$

at $x = \frac{\pi}{2}$, $a + b.0 = 0$.

Hence $a = b = 0$. Thus $\{f, g\}$ is l.i. (1/2) presentation

b) Since $\sin(x+y) = \sin x \cos y + \sin y \cos x$, $\forall x, y \in \mathbb{R}$, we have ($\frac{1}{2}$)

$$h(x) = \sin(x + \pi/6) = \sin x . \cos \pi/6 + \sin \pi/6 . \cos x, \quad \forall x \in \mathbb{R}$$

(1) $\longrightarrow$ $= \frac{\sqrt{3}}{2} . \sin x + \frac{1}{2} \cos x$

Hence $h = \frac{\sqrt{3}}{2} . f + \frac{1}{2} g$, so $\{f, g, h\}$ is l.d.

($\frac{1}{2}$) presentation

c) Since $h \in \text{span}\{f, g\}$, $\text{span}\{f, g, h\} = \text{span}\{f, g\}$. (1)

Thus, $\{f, g\}$ spans $\text{span}\{f, g, h\}$. Moreover, from (a), we know $\{f, g\}$ is l.i. Hence $\{f, g\}$ is a basis for $\text{span}\{f, g, h\}$ and so $\dim \text{span}\{f, g, h\} = 2$. (1)

($\frac{1}{2}$)

($\frac{1}{2}$) presentation

2. Let $T : V \to W$ be a linear map and $\{v_1, \ldots, v_n\}$ a subset of V.

4 a) If $\{v_1, \ldots, v_n\}$ is linearly dependent in V, show that $\{T(v_1), \ldots, T(v_n)\}$ is linearly dependent in W.

3 b) If $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent in W, show that $\{v_1, \ldots, v_n\}$ is linearly independent in V.

a) Since $\{v_1, .., v_n\}$ is l.d., $\exists\, c_1, .., c_n \in F$, with $(c_1, .., c_n) \neq 0$, s.t. $\underline{\sum_{i=1}^{n} c_i v_i = 0}$. (1)

Since $T(0) = 0$, we have $\underline{T\left(\sum_{i=1}^{n} c_i v_i\right) = 0}$. As T is linear, this is equivalent to $\sum_{i=1}^{n} c_i T(v_i) = 0$. ($\frac{1}{2}$)

Recalling that $(c_1, .., c_n) \neq 0$, we see that $\{Tv_1, .., Tv_n\}$ is also l.d.

(1) - presentation

b) This is simply the contrapositive of (a): If $\{Tv_1, .., Tv_n\}$ is l.i., then $\{v_1, .., v_n\}$ cannot be dependent, since by (a), this would imply that $\{Tv_1, .., Tv_n\}$ is l.d. Hence $\{v_1, .., v_n\}$ is also l.i.

(2) - correct argument

(1) - presentation

3. Let $A = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$ and define a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$T(v) = Av, \quad x \in \mathbf{R}^3.$$

$1\frac{1}{2}$ a) Find a basis for $\ker T$.

$1\frac{1}{2}$ b) Find a basis for $\operatorname{im} T$.

2 c) Is $\operatorname{im} T$ a subspace of $\ker T$?

2 d) Is $\ker T$ a subspace of $\operatorname{im} T$?

a) $\ker T = \ker A = \ker \begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix} = \ker \begin{bmatrix} 1 & 4 & 2 \\ 0 & -2 & -2 \\ 0 & 5 & 5 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$= \left\{ \begin{bmatrix} 2s \\ -s \\ s \end{bmatrix} \middle| s \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$. Since $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \neq 0$, $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ is

l.i. and hence $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\ker T$. 1 – justification

1/2

b) From (a), we see that $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis of $\operatorname{im} T = \operatorname{col} A$.

1 (1st 2 cols – where leading ones occur). 1/2

c) $\operatorname{im} T$ cannot be a subspace of $\ker T$ because $\dim \operatorname{im} T = 2 > \dim \ker T = 1$. 2 (1, 1) correct and justified

(correct ans any justn: 0/2)

d) $\ker T$ will be a s.s. of $\operatorname{Im} T$ iff $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$, so we

consider the system $\left[\begin{array}{cc|c} 1 & 4 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & -2 & -3 \\ 0 & 5 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & -\frac{9}{2} \end{array}\right]$,

which is inconsistent. Hence, $\ker T$ is <u>not</u> a subspace of $\operatorname{im} T$.

2 (1, 1) – correct & justified

4. Define a function $f : \mathbf{R}^3 \to \mathbf{R}$ by $f(x, y, z) = x + y + z$.

2 a) Prove that f is a linear form on $\mathbf{R}^3$.

2 b) Find a basis for $\ker f$, the kernel of f.

3 c) Can you find a subspace V of $\mathbf{R}^3$ such that $\ker f \oplus V = \mathbf{R}^3$?

① knowing what to do

① doing it 'well'

a) Note that $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; Since f is multiplication by a matrix, it is linear.

b) $\ker f = \ker \begin{array}{ccc} & s & t \\ \left[1\right. & 1 & \left.1\right] \end{array} = \left\{ \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$

① - knowing what ker f is

$= \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. This algorithm produces l.i. vectors, so

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\ker f$. ① ① - correct basis w/ some justn

c) If we take $v \notin \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, then

① - "yes" + some just

② - "yes" + correct just

③ = ② + ① presentation

$\left\{ v, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ will be a basis of $\mathbb{R}^3$, since it is l.i., and contains $3 = \dim \mathbb{R}^3$ vectors. We choose $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We know $v \notin \ker f$ since $f(v) = 1 \neq 0$. Now set $V = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$. Since

$\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\in V}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{both in } \ker f} \right\}$ spans $\mathbb{R}^3$, we know $V + \ker f = \mathbb{R}^3$.

Moreover, if $w \in V \cap \ker f$, then $w = \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}$, and $f\left(\begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}\right) = \lambda = 0$. Hence $w = 0$ and so $V \cap \ker f = \{0\}$. Hence, $\ker f \oplus V = \mathbb{R}^3$.

5. Let $U = \{p \in \mathcal{P}_2(\mathbf{R}) \mid p(1) = p(0) = 0\}$.

$2\frac{1}{2}$ a) Find a basis for U, and give the dimension of U.

2 b) Extend your basis of U in (a) to a basis of $\mathcal{P}_2(\mathbf{R})$.

$2\frac{1}{2}$ c) Can you find a subspace V of $\mathcal{P}_2(\mathbf{R})$ such that $U \oplus V = \mathcal{P}_2(\mathbf{R})$?

a) By the remainder theorem, $p \in U \iff p(t) = t(t-1)q(t)$ where $\deg q = \deg p - 2$. Since $\deg p \leq 2$, $\deg q = 0$. Hence $p(t) = a\,t(t-1)$ for some $a \in \mathbb{R}$. Thus $U = \operatorname{span}\{t(t-1)\}$.

Moreover, $t(t-1) \neq 0$, so $\{t(t-1)\}$ is a basis of U. Hence $\dim U = 1$ ($\frac{1}{2}$)

(1) - correct basis

(1) - just^n

b) We claim that $\{1, t, t(t-1)\}$ is a basis of $\mathcal{P}_2(\mathbb{R})$. Since $\dim \mathcal{P}_2(\mathbb{R}) = 3$, it suffices to show that this set is l.i.

So suppose $a \cdot 1 + bt + c \cdot t(t-1) = 0$, $\forall t \in \mathbb{R}$. At

(1) - correct ext^n

(1) - just^n

$t = 0$, we obtain $a = 0$

$t = 1$, " $a + b = 0$

$t = -1$, " $a - b + 2c = 0$.

It is easy to see that the only sol^n of this system is $a = b = c = 0$. Hence $\{1, t, t(t-1)\}$ is a basis of $\mathcal{P}_2(\mathbb{R})$ extending the basis of U in (a).

c). We set $V = \operatorname{span}\{1, t\}$. Then $V + U = \mathcal{P}_2(\mathbb{R})$ because $\{1, t, t(t-1)\}$ spans $\mathcal{P}_2(\mathbb{R})$. Moreover, if $w \in V \cap U$, then $w = a + bt = c' t(t-1)$ for some scalars a, b, c'. But then $a + bt - c' t(t-1) = 0$. However, we know $\{1, t, t(t-1)\}$ is l.i., and so $a = b = c' = 0$. In particular, $w = 0$. Hence $V \cap U = \{0\}$, and so $U \oplus V = \mathcal{P}_2(\mathbb{R})$.

(1) - yes + some just^n or (2) good just^n or ($2\frac{1}{2}$) " + well-written

6. (Bonus) Let V be a finite dimensional vector space over the field $\mathbf{C}$ of complex numbers. With the same addition, and multiplication by real scalars defined by

$3\frac{1}{2}$

$$kv = (k + 0i) \quad \forall k \in \mathbf{R},\ \forall v \in V,$$

V becomes a vector space over the field $\mathbf{R}$ of real numbers. Prove that V is finite dimensional over the field $\mathbf{R}$, and that $\dim_{\mathbf{R}} V = 2 \dim_{\mathbf{C}} V$.

Let $\mathcal{C} = \{v_1, \ldots, v_n\}$ be a basis of V over $\mathbb{C}$.

We claim that $\mathcal{B} = \{v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n\}$ ($2n$ vectors in all) forms a basis of V over $\mathbb{R}$.

To see that $\mathcal{B}$ is l.i., suppose there are real scalars $c_1, d_1, c_2, d_2, \ldots, c_n, d_n$ such that $\sum_{j=1}^{n} (c_j v_j + d_j i v_j) = 0$. Then, we have

$\sum_{j=1}^{n} (c_j + id_j) v_j = 0$. But $\{v_1, \ldots, v_n\}$ is l.i. over $\mathbb{C}$, so $c_j + id_j = 0$, for $j = 1, \ldots, n$. Hence $c_1 = d_1 = c_2 = d_2 = \cdots = c_n = d_n = 0$

choice ($\frac{1}{2}$)
l.i. (1)
spans (1)
presentation (1)

To see that $\mathcal{B}$ spans V over $\mathbb{R}$, let $v \in V$ be any vector. Because $\mathcal{C}$ is a basis over $\mathbb{C}$, there are complex scalars $a_1, \ldots, a_n$ s.t. $v = \sum_{j=1}^{n} a_j v_j$. Now we can write $a_j = c_j + id_j$ for real numbers c_j, d_j, $j = 1, \ldots, n$. Thus,

$$v = \sum_{j=1}^{n} (c_j + id_j) v_j = \sum_{j=1}^{n} (c_j v_j + d_j (iv_j))$$

Hence, $\mathcal{B}$ spans V over $\mathbb{R}$. Thus $\mathcal{B}$ is a basis of V over $\mathbb{R}$, and so $\dim_{\mathbb{R}} V = 2n = 2 \dim_{\mathbb{C}} V$, as required.