* * * SOLUTIONS * * *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences
Midterm Test
MATA37 - Calculus II for Mathematical Sciences

Examiner: R. Grinnell

Date: June 14, 2014
Time: 5:00 pm
Duration: 120 minutes

(Print in CAPITALS) LAST NAME: ______________________

(Print) Given Name(s): ______________________

Student Number: ______________________

Signature: Solutions + Basic Stats

Circle the name of your Teaching Assistant and Tutorial Number:

Yuri CHER 2 3

1 Eric CORLETT

Aaron CHOW 5

4 Mikhail GUDIM

Read these instructions:

1. This test has 10 pages. It is your responsibility to check at the beginning of the test that all of these 10 pages are included.
2. If you need extra answer space for any question, use the back of a page or the last page. Clearly indicate the location of your continuing work.
3. The following are forbidden at your workspace: calculators, any other kind of electronic aid or device (e.g. cell/smart phones, i-pads, i-phones, etc.), scrap paper, food, textbooks, bags, pencil/pen carrying cases, drinks in a paper cup or box or similar container that has a removable label.
4. Cell/smart/i-phones must be turned off and left at the front of the test room.
5. You are encouraged to write in pen or other ink, not pencil. If any part of your test is written in pencil, then you will be denied any re-grading opportunity.

* SOLUTIONS *

Print letters for the True/False Questions in these boxes.

1	2	3	4	5	6	7
T	F	F	F	F	T	F

Do not write anything in the boxes below.

2 points ⟺ all info requested on page 1 is present.

Info.	Part A
2	21

Part B

1	2	3	4	5	6
16	14	13	11	11	12

Total
100

Part A - True/False Questions For each of the following assertions answer only with

T = "Always True" **F** = "False" (i.e. Not Always True) **O** = "You Do Not Know"

Each right answer earns 3 points. Each wrong or ambiguous answer earns -1 point. Each O earns 0 points. A small workspace is provided for your rough work. **Put your answers in the boxes at the top of page 2**

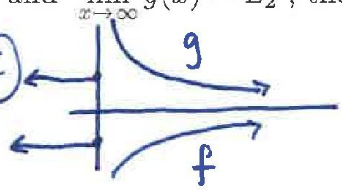
1. Every non-empty subset of the positive real numbers has an infimum.

T If $A \subseteq (0,\infty)$ and $A \neq \phi$, then 0 is a lower bound....
Use IA.

2. If x and y are real numbers, then $\left| x - y \right| \leq \left| |x| - |y| \right|$

F Example: let $x = -2$ and $y = 2$
$|x-y| = 4$ $\big| |x| - |y| \big| = |2-2| = 0$.

3. Assume f and g are functions defined on $\mathbf{R}$ such that $f(x) < g(x)$ for all x. If $\lim_{x\to\infty} f(x) = L_1$ and $\lim_{x\to\infty} g(x) = L_2$, then $L_1 < L_2$.

F 

Example: $f(x) = -\frac{1}{x}$ $g(x) = \frac{1}{x}$ if $x > 0$;
$f(x) = -1$, $g(x) = 1$ if $x \leq 0$. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x)$
$f(x) < g(x)$ $\forall x \in \mathbb{R}$

4. Let h be a function defined for all real numbers and assume $\lim_{x\to a} h(x) = l$. We can deduce that given any $\epsilon > 0$ there exists a $\delta > 0$ such that $|h(x) - l| < \epsilon$ whenever $|x - a| < \delta$.

F We need $0 < |x-a| < \delta$. Example: $h(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$
$\lim_{x\to 0} h(x) = 0$ but $|h(0) - 0| = 1$

5. Assume functions f and g are defined for all real numbers. Then f and g are continuous at a point c if and only if the sum function $f + g$ is continuous at c.

F Example: $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ $g(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$

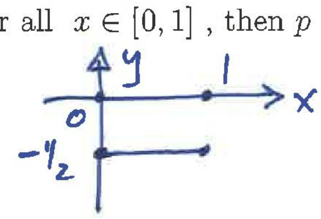
f nor g is cts @ 0 yet f+g is cts at 0.

6. The function $\varphi(x) = ln\left(\dfrac{x}{x+1}\right)$ is continuous on the interval $(-\infty, -1)$.

T φ is cts $\Leftrightarrow \frac{x}{x+1} > 0$ $\Big\}$ $\therefore \frac{x}{x+1} > 0$
When $x < -1 \Rightarrow x+1 < 0$ $\therefore \varphi$ is cts on $(-\infty, -1)$.

7. If a polynomial $y = p(x)$ has the property that $|p(x)|^2 \leq |p(x)|$ for all $x \in [0,1]$, then p has a fixed point in $[0,1]$.

F Example: $p(x) = -\frac{1}{2}$ $\forall x \in [0,1]$



p has no f.p. in $[0,1]$ But $|p(x)|^2 = \frac{1}{4}$
$|p(x)| = \frac{1}{2}$.

*** DID YOU PUT THE ANSWERS IN THE BOXES ON PAGE 2? ***

Part B - Full Solution Questions Put your answers and solutions in the space provided. Full points will be awarded for your solutions if and only if they are correct, complete, and show sufficient relevant concepts form MATA37.

1. (a) Let h be a function defined on a set of real numbers D. Give the precise definition of what it means to say that $\lim_{x \to a} h(x)$ exists.

[5 points]

$\lim_{x \to a} h(x)$ exists $\Leftrightarrow$ $\exists \ell \in \mathbb{R}$ such that $\forall \varepsilon > 0$ $\exists \delta > 0$ so that $|h(x) - \ell| < \varepsilon$ whenever $x \in D$ and $0 < |x - a| < \delta$.

(b) Give a complete and accurate proof that $\lim_{x \to 4} \dfrac{6}{\sqrt{x+5}}$ exists. [11 points]

Claim that $\lim_{x \to 4} \dfrac{6}{\sqrt{x+5}} = \dfrac{6}{\sqrt{4+5}} = \dfrac{6}{3} = 2$.

Proof: We have the following calculation for $x > 0$:

$$\left| \frac{6}{\sqrt{x+5}} - 2 \right| = \left| \frac{6 - 2\sqrt{x+5}}{\sqrt{x+5}} \right| = \left| \frac{(6 - 2\sqrt{x+5})(6 + 2\sqrt{x+5})}{\sqrt{x+5}\,(6 + 2\sqrt{x+5})} \right|$$

$$= \frac{|36 - 4x - 20|}{\sqrt{x+5}\,(6 + 2\sqrt{x+5})} = \frac{4|x-4|}{\sqrt{x+5}\,(6 + 2\sqrt{x+5})} < \frac{4|x-4|}{2(6 + 2(2))}$$

$$= \frac{|x-4|}{5} < |x-4|$$

($\because x > 0 \therefore \sqrt{x+5} > 2$)

Given $\varepsilon > 0$ we let $\delta = \min(\varepsilon, 4)$. If $0 < |x-4| < \delta$ then $x > 0$ and the calculations above show that $\left| \dfrac{6}{\sqrt{x+5}} - 2 \right| < \varepsilon$.

$\therefore \lim_{x \to 4} \dfrac{6}{\sqrt{x+5}} = 2$ so the limit exists. $\blacksquare$

2. Assume f and g are functions defined on $\mathbf{R}$ and continuous at a point a. Give a complete and accurate proof that the product function fg is also continuous at a.

[14 points]

Let $\varepsilon > 0$ be given. We show $\exists \delta > 0$ so that if $|x-a| < \delta$ then

$$|f(x)\,g(x) - f(a)\,g(a)| < \varepsilon.$$

Consider these "estimates":

$$|f(x)g(x) - f(a)g(a)|$$
$$= |f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)|$$
$$\le |f(x)g(x) - f(a)g(x)| + |f(a)g(x) - f(a)g(a)|$$
$$= |g(x)|\,|f(x) - f(a)| + |f(a)|\,|g(x) - g(a)|.$$

$\because$ g is cts @ a $\Rightarrow \exists \delta_1 > 0 \ni$: if $|x-a| < \delta_1$ then $|g(x) - g(a)| < 1$ so $|g(x)| < 1 + |g(a)|$.

$\because$ f is cts @ a $\Rightarrow \exists \delta_2 > 0 \ni$: if $|x-a| < \delta_2$ then
$$|f(x) - f(a)| < \frac{\varepsilon}{2(1+|g(a)|)}.$$

$\because$ g is cts @ a $\Rightarrow \exists \delta_3 > 0 \ni$: if $|x-a| < \delta_3$ then
$$|g(x) - g(a)| < \frac{\varepsilon}{2(1+|f(a)|)}.$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $|x-a| < \delta$, then our work above shows that

$$|f(x)\,g(x) - f(a)\,g(a)|$$
$$< (1 + |g(a)|)\frac{\varepsilon}{2(1+|g(a)|)} + |f(a)|\frac{\varepsilon}{2(1+|f(a)|)} < \varepsilon$$

$\therefore$ fg is continuous @ a. ▨

3. (a) Give a complete and accurate proof that $\alpha = \frac{3}{7} + \frac{\sqrt{5}}{2}$ is algebraic. [6 points]

$14\alpha = 6 + 7\sqrt{5}$

$14\alpha - 6 = 7\sqrt{5}$

$\therefore (14\alpha - 6)^2 = (7\sqrt{5})^2$

$196\alpha^2 - 168\alpha + 36 = 245$

$\therefore 196\alpha^2 - 168\alpha - 209 = 0$

We let

$p(x) = 196x^2 - 168x - 209$

- p has integer coefficients.
- $p(\alpha) = 0$

$\therefore \alpha$ is algebraic.

(b) Let $\beta \in \mathbf{R}$ and define $A = \{x \in \mathbf{Q} : x < \beta\}$. Justify why $\sup(A)$ exists and show that $\sup(A) = \beta$. You may use the fact that the set $\mathbf{Q}$ of rational numbers is dense. [7 points]

$\because \mathbb{Q}$ is dense $\Rightarrow \exists\, q \in \mathbb{Q} \ni: \beta - 1 < q < \beta$.

$\therefore q \in A$ so $A \neq \phi$. It is clear that $A \subset \mathbb{R}$.

Also, if $x \in A \Rightarrow x < \beta$, so β is an upper bound of A. $\therefore$ by SA, $\sup(A)$ exists.

$\because \beta$ is an upper bound of $A \Rightarrow \sup(A) \leq \beta$.

Suppose $\sup(A) < \beta$. $\because \mathbb{Q}$ is dense,

$\exists\, r \in \mathbb{Q} \ni: \sup(A) < r < \beta$. Thus $r \in A$.

But this contradicts $\sup(A)$ being an upper bound of A. $\therefore \sup(A) \not< \beta$

and $\sup(A) \leq \beta$, thus $\sup(A) = \beta$.

4. (a) State the triangle inequality. [2 points]

$\forall x, y \in \mathbb{R}$, $|x+y| \le |x| + |y|$.

(b) Use the Principle of Mathematical Induction (PMI) to prove the following:

For each natural number n, given any set of n different real numbers $x_1, x_2, x_3, \ldots, x_n$, we have that $\left|\sum_{k=1}^{n} x_k\right| \le nB$ where $B = max\left\{|x_k| \ : \ k = 1,\ 2,\ 3,\ \ldots, n\right\}$.

[9 points]

Let $n=1$ so we have 1 number : x_1

$\left|\sum_{k=1}^{1} x_1\right| = |x_1| \le 1B$ where $B = \max\{|x_k| : k=1\}$
$= |x_1|$.

$\therefore$ statement is true when $n=1$.

Assume the assertion above is true for some $n \in \mathbb{N}$.

For this n, let us be given $n+1$ different real numbers $x_1, x_2, x_3, \ldots, x_n, x_{n+1}$.

Let $M_1 = \max\{|x_k| : k = 1, 2, \ldots, n\}$ and

$M = \max\{M_1, |x_{n+1}|\}$

$= \max\{|x_k| : k = 1, 2, 3, \ldots, n, n+1\}$

We have that

$$\left|\sum_{k=1}^{n+1} x_k\right| = \left|\sum_{k=1}^{n} x_k + x_{n+1}\right|$$

$$\le \left|\sum_{k=1}^{n} x_k\right| + |x_{n+1}| \le nM_1 + |x_{n+1}|$$

$$\le nM + |x_{n+1}| \le nM + M = (n+1)M.$$

That shows the assertion is true for $n+1$.

$\therefore$ by the PMI the assertion is true $\forall n \in \mathbb{N}$. ▨

5. The parts of this question are independent of each other.

(a) Let E be a set of real numbers and let $h : E \to \mathbf{R}$ be a function that is continuous and positive at a point c. Prove there is an open interval I containing c such that $h(x) > 0$ for all $x \in E \cap I$.

[7 points]

We have that $h(c) > 0$ and h is cts at c. $\therefore$ with $\varepsilon = \frac{h(c)}{2} > 0$, $\exists \delta > 0$ so that $|h(x) - h(c)| < \frac{h(c)}{2}$ for all $x \in E$ such that $|x - c| < \delta$.

Now $|x-c| < \delta \Leftrightarrow c - \delta < x < c + \delta$
$\Leftrightarrow x \in (c-\delta, c+\delta) = I$

$\therefore$ if $x \in E \cap I$, $|h(x) - h(c)| < \frac{h(c)}{2}$

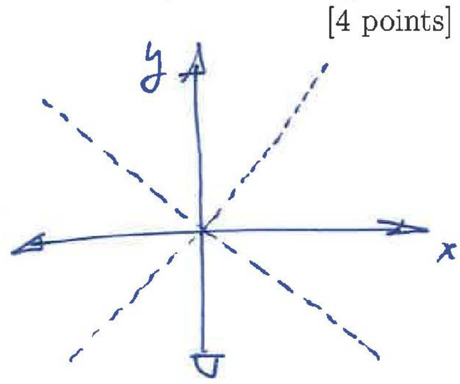
$\Rightarrow -\frac{h(c)}{2} < h(x) - h(c)$

$\Rightarrow \frac{h(c)}{2} < h(x)$ so $h(x) > \frac{h(c)}{2} > 0$.

(b) Give an example of functions f and g that are both continuous at 0, but the composition function $f \circ g$ is not continuous at 0.

[4 points]

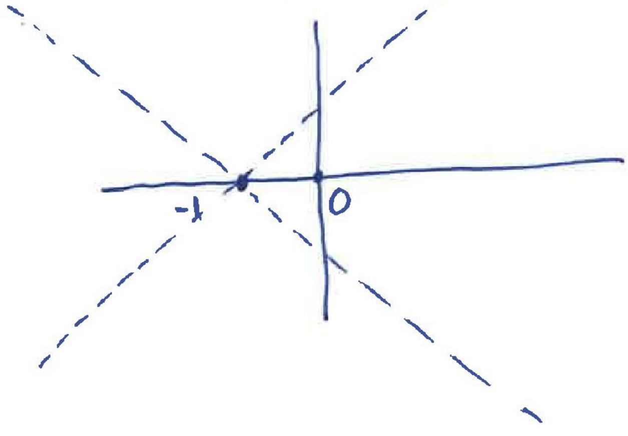
Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{I} \end{cases}$



f is cts at 0 (and only @ 0)

Let $g(x) = x+1$ which is cts at every real x

$(f \circ g)(x) = f(g(x)) = f(x+1) = \begin{cases} x+1 & \text{if } x+1 \in \mathbb{Q} \\ -(x+1) & \text{if } x+1 \in \mathbb{I} \end{cases}$



$f \circ g$ is only cts at -1, so not cts at 0.

[There are "many" other examples].

6. (a) Give a precise statement of the Intermediate Value Theorem (aka IVT). [4 points]

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ where $a, b \in \mathbb{R}$ $a < b$. If k is between $f(a)$ and $f(b)$ (ie $f(a) \leq k \leq f(b)$ or $f(b) \leq k \leq f(a)$) then $\exists\, c \in [a,b]$ $\ni$: $f(c) = k$.

(b) Assume $\psi : [a,b] \to \mathbf{R}$ is continuous on the interval $[a,b]$. Let m and n be positive real numbers. Prove there is some $c \in [a,b]$ such that $\psi(c) = \dfrac{m\psi(a) + n\psi(b)}{m+n}$.

[8 points]

Let $g(x) = \dfrac{m\psi(a) + n\psi(b)}{m+n} - \psi(x)$

where $x \in [a,b]$. $\because$ ψ is cts on $[a,b]$, g is also cts on $[a,b]$.

$$g(a) = \frac{m\psi(a) + n\psi(b)}{m+n} - \psi(a)$$

$$= \frac{m\psi(a) + n\psi(b) - m\psi(a) - n\psi(a)}{m+n}$$

$$= \frac{n(\psi(b) - \psi(a))}{m+n}$$

$$g(b) = \frac{m\psi(a) + n\psi(b)}{m+n} - \psi(b)$$

$$= \frac{m\psi(a) + n\psi(b) - m\psi(b) - n\psi(b)}{m+n}$$

$$= \frac{m(\psi(a) - \psi(b))}{m+n}$$

If $\psi(a) = \psi(b)$ then we can let $c = a$ (or $c = b$).
If $\psi(a) \neq \psi(b)$ then $g(a)$ and $g(b)$ have opposite signs. By OST $\exists\, c \in [a,b]$ so that $g(c) = 0$.
For this c, $\psi(c) = \dfrac{m\psi(a) + n\psi(b)}{m+n}$. $\blacksquare$

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Basic Statistics (*)

N = # of students who wrote test = 122

$\bar{x}$ = 48.73% = average

90's — 0.0%

80's — 0.8%

70's — 4.1%

60's — 17.2%

50's — 23.8%

40's — 39%

30's — 14.6%

20's — 6.6%

10's — 0.8%

This is very poor for UTSC math/cs students.

% of the 122 students in each 10% bracket

≈ 5% of 122 students got ≥ 70%

≈ 22% " " " " ≥ 60%

≈ 46% " " " " ≥ 50% (pass)

(*) before any re-grading or contributions as a result of the "Special Quiz".