University of Waterloo

AMath 250 – Introduction to Differential Equations

Midterm Examination Spring 2011

Monday, June 20th. Time: 6:30 - 8:00 p.m.

Name (print): SOLUTIONS

I.D. Number: ______________________

Signature: ______________________

Closed book.
Faculty-approved calculators permitted.

Note: Your grade will be influenced by the clarity of your solutions.

Question	Marks Available	Marks Awarded
1	14	
2	9	
3ab	11	
3cd	12	
4	14	
Total	60	

[14] (8+6) 1. Solve the differential equation $\frac{dy}{dx} = y - x^2$, and sketch the family of solutions. Your sketch should include any exceptional solutions and a representative selection of the others (if you have a constant C in your solution, then you should probably show what the curves look like for $C > 0$ and for $C < 0$). It should also show where any critical points occur.

The equation is linear, with a constant coefficient: $\frac{dy}{dx} - y = -x^2$.

The homogeneous solution (or "complementary function") is $y_h = Ce^{-x}$, by inspection. (2)

For a particular solution, we try

$y_p = Ax^2 + Bx + C$, (2)

so $y_p' = 2Ax + B$.

Plugging this into the DE yields $2Ax + B - Ax^2 - Bx - C = -x^2$,

from which we deduce that $A = 1$,

$2A - B = 0$,

and $B - C = 0$. (2)

That is, $A = 1$, $B = 2$, and $C = 2$, so $y_p = x^2 + 2x + 2$.

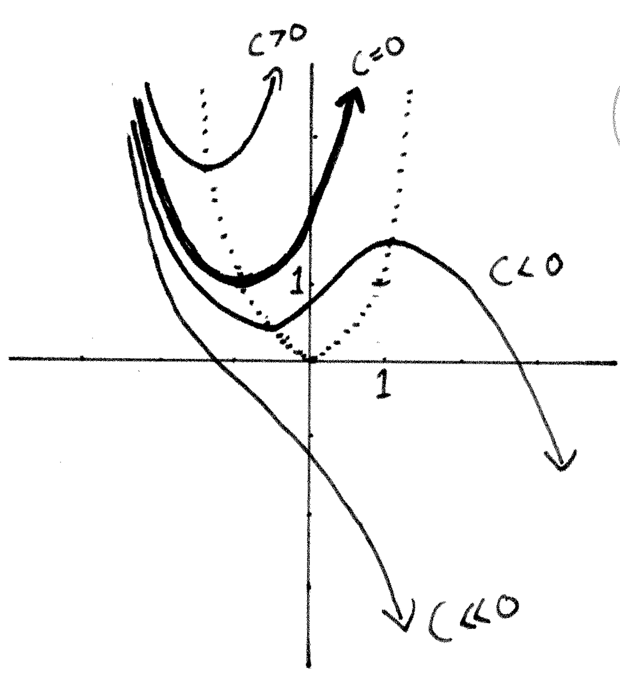
Hence $y = Ce^{-x} + x^2 + 2x + 2$. (2) for $y = y_h + y_p$.

Exceptional Solution: $y = x^2 + 2x + 2 = (x+1)^2 + 1$ (1)

Horizontal Isocline: $y' = 0 \Rightarrow y = x^2$. (1)

As $x \to \infty$, $Ce^{-x} \to 0$.

As $x \to -\infty$, $Ce^{-x} \to \infty$ if $C > 0$, and $Ce^{-x} \to -\infty$ if $C < 0$.



(4) (1) for sketch of ex'l sol'n
(1) for actually having $y' = 0$ on $y = x^2$
(2) for rest

9

[11] 2. Solve the initial value problem:

$$x\frac{dy}{dx} + (x+1)y = e^{-x}, \qquad y(1) = 0.$$

This is linear. In standard form it is $\frac{dy}{dx} + \left(\frac{x+1}{x}\right)y = \frac{e^{-x}}{x}$. 1

Hence the integrating factor is 2

$I(x) = e^{\int(1+\frac{1}{x})dx} = e^{x+\ln x} = xe^{x}$. 2

Incorporating this gives $xe^{x}\frac{dy}{dx} + (x+1)e^{x}y = 1$

i.e. $\frac{d}{dx}\left(xye^{x}\right) = 1$, 2

so $xye^{x} = x + C$. 1

$y(1) = 0 \Rightarrow C = -1$, so $y = \frac{x-1}{xe^{x}} = e^{-x}\left(1 - \frac{1}{x}\right)$ 1

3. In class you were introduced to the Malthusian model of population growth (the exponential growth model), and also to the logistic model (which incorporates the idea of a maximum sustainable population). In the 19th century an English actuary suggested an alternative to the logistic model, now known as the Gompertz equation:

$$\frac{dy}{dt} = ry \ln\left(\frac{K}{y}\right).$$

The constant r is a growth constant (as used in the Mathusian and logistic models), and the constant K is the carrying capacity (as in the logistic model).

[7] a) Rewrite the equation in terms of dimensionless variables.

Let $[y] = P$ (population).

We also have $[t] = T$.

Meanwhile, $[r] = T^{-1}$, and $[K] = P$, so we define $t_c = \frac{1}{r}$, and $y_c = K$. 2

We can then define dimensionless variables as

$\tau = rt$, and $\eta = \frac{y}{K}$. 2 (OK to go straight to here for 4 marks)

Now, $\frac{dy}{dt} = \frac{dy}{d\eta}\frac{d\eta}{d\tau}\frac{d\tau}{dt} = K\frac{d\eta}{d\tau}r$, 2

so the equation becomes $rK\frac{d\eta}{d\tau} = rK\eta \ln\left(\frac{1}{\eta}\right)$

i.e. $\frac{d\eta}{d\tau} = -\eta \ln \eta$. 1

4 [3] b) Show that any curve crossing the line $y = \frac{K}{e}$ will have an inflection point there.

If $\frac{d\eta}{d\tau} = -\eta \ln \eta$, then $\frac{d^2\eta}{d\tau^2} = -\ln\eta - 1$. 2

Hence $\frac{d^2\eta}{d\tau^2} = 0$ when $\ln \eta = -1$,

i.e. $\eta = e^{-1}$, 1

and η'' clearly changes sign there.

Finally, $\eta = e^{-1}$ means $y = Ke^{-1}$. 1

[7] 9 c) Assuming that $y(0) = P_0$, find $y(t)$. You may assume that $P_0 \le K$, and that $y(t)$ remains $\le K$ for all time.

In terms of η and τ, the initial condition is $K\eta(0) = P_0$,

i.e. $\eta(0) = P_0/K$.

Now, $\frac{d\eta}{d\tau} = -\eta \ln \eta$ is separable:

$$\int \frac{d\eta}{\eta \ln \eta} = \int -d\tau$$ 2

$$\boxed{\begin{array}{l} \text{let } u = \ln \eta \\ du = \frac{1}{\eta} d\eta \end{array}}$$

$$\Rightarrow \int \frac{du}{u} = -\tau + C$$

$$\Rightarrow \ln|u| = -\tau + C$$

$$\Rightarrow \ln|\ln \eta| = -\tau + C$$ 2

The IC gives $\ln|\ln \frac{P_0}{K}| = C$,

so $\ln|\ln \eta| = \ln|\ln \frac{P_0}{K}| - \tau$ 2

$$\Rightarrow |\ln \eta| = e^{\ln|\ln \frac{P_0}{K}|} e^{-\tau}$$

$$= |\ln \tfrac{P_0}{K}| e^{-\tau}$$ 2

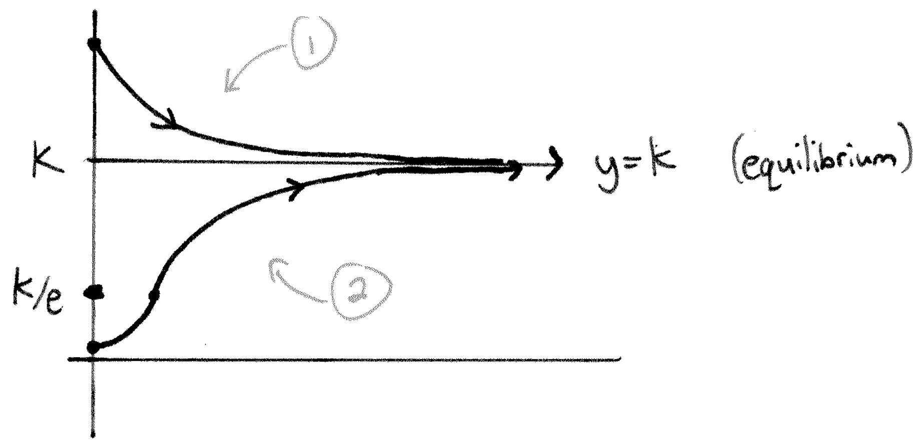
$$\Rightarrow -\ln \eta = -\ln\left(\tfrac{P_0}{K}\right) e^{-\tau}$$ (since $\eta = \frac{y}{K}$ and $\frac{P_0}{K}$ are both less than 1) $*$

$$\Rightarrow \eta = e^{\ln\left(\frac{P_0}{K}\right) e^{-\tau}}$$

i.e. $\eta = \left(\frac{P_0}{K}\right)^{e^{-\tau}}$ ($\frac{P_0}{K}$ raised to the power of $e^{-\tau}$)

Hence $y = K\left(\frac{P_0}{K}\right)^{e^{-rt}}$ 1 (returning to original variables)

[6] [3] d) Sketch a typical solution with $P_0 << K$. What should a solution with $P_0 > K$ look like? ~~Justify your answer (based on mathematics, not just on intuition).~~

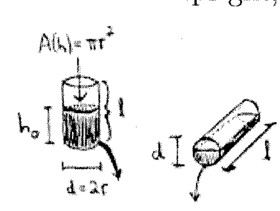


Comments:

- The Gompertz model gives solutions which are similar to the solutions to the logistic model, except that the growth rate peaks when $y = \frac{K}{e}$ instead of at $\frac{K}{2}$, and gives lower populations at all times.

- You can see what happens if $P_0 > K$ in either of two ways: ① Repeat part (c) with $P_0 > K$; the only difference will be that no negatives appear in line (✱) on the previous page. ② Observe simply that $\frac{dy}{dt} < 0$ (from the DE itself), and there are no inflection points (from part b), and the solutions cannot intersect with $y = K$.

4. On your third assignment you showed that the time required for a cylinder full of water to drain is $t_{\text{vert}} = \frac{2\pi r^2 \sqrt{l}}{k}$ if the cylinder stands upright, while it is



$t_{\text{horiz}} = \frac{4ld^{3/2}}{3k} = \frac{2^{5/2} r^{3/2} l}{3k}$ if it lies on its side.

[2] a) What are the dimensions of k? (Show how you can *determine* this from one of the results above.)

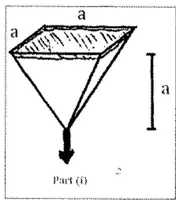
$$[k] = \frac{[2\pi][r]^2[l]^{1/2}}{[t_{\text{vert}}]} = \frac{1 \cdot L^2 \cdot L^{1/2}}{T} = L^{5/2} T^{-1}$$

(2)

b) Now consider a container in the form of an inverted pyramid. Suppose we fill this with water and drill a similar hole in the bottom vertex. What does Buckingham's Pi Theorem allow you to say about the drainage time T in each of the following cases? Explain your logic.

Hint: T should depend *only* upon k and the relevant length parameter(s).

[5] 6 i) Pyramid as illustrated, with only one length parameter, a.



$[T] = T$

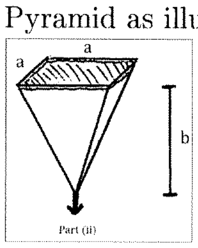
$[k] = L^{5/2} T^{-1}$

$[a] = L$.

② for 3-2=1
② for Π
② for conclusion

We have 3 ~~variables~~ quantities and 2 dimensions, so there is one dimensionless quantity: $\Pi = \frac{kT}{a^{5/2}}$. Conclusion: $\Pi = C$, or $T = \frac{Ca^{5/2}}{k}$.

[3] [4] ii) Pyramid as illustrated, with two length parameters, a and b.

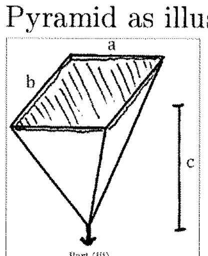


① for 4-2=2
① for Π_2
② for conclusion

Here there are 4 quantities, and still only 2 dimensions. Therefore there must be a second dimensionless quantity: $\Pi_2 = \frac{a}{b}$.

Conclusion: $\Pi_1 = f(\Pi_2)$, i.e. $\frac{kT}{a^{5/2}} = f\left(\frac{a}{b}\right)$, so $T = \frac{a^{5/2}}{k} f\left(\frac{a}{b}\right)$

[2] iii) Pyramid as illustrated, with three length parameters, a, b, and c.



Now it's $5-2=3$, and a 3rd independent dimensionless variable is $\Pi_3 = \frac{a}{c}$.

Conclusion: $\Pi_1 = f(\Pi_2, \Pi_3)$,

i.e. $T = \frac{a^{5/2}}{k} f\left(\frac{a}{b}, \frac{a}{c}\right)$.

5-2=3 and Π_3 should be obvious if (ii) was understood, so just ② for conclusion.