MCV4U-A

Unit 1

Rates of Change and Derivatives

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Introduction

Calculus is a branch of mathematics that is thought of as a gateway to higher-level mathematics. The origins of calculus can be traced to ancient Egypt when ideas involving calculus were used to find the volume of non-standard shapes. Modern calculus has its foundation with the work of Isaac Newton and Gottfried Wilhelm Leibniz in the seventeenth century. Calculus has many applications in physics, economics, and astronomy.

In Unit 1, you will start your journey into calculus by looking at the rate of change. It is important to study how changes affect your daily life. For example, the price a manufacturer charges for a product depends on a number of variables, such as the cost of raw material, the cost of labour involved, and so on. As the value of these variables change, the total cost of the product changes. Calculus helps quantify this change.

Overall Expectations

After completing this unit, you will be able to

- demonstrate an understanding of rate of change by making connections between average rate of change over an interval and instantaneous rate of change at a point, using the slopes of secants and tangents and the concept of the limit
- graph the derivatives of polynomial functions, and make connections between the numeric, graphical, and algebraic representations of a function and its derivative
- apply the rules of differentiation

Review

Slope of a line through A(x_1, y_1) and B(x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$.

The slope/intercept form of a line is $y = mx + b$, where m is the slope and b is the y-intercept.

Quadratic formula: The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Changing a quadratic function from standard form $y = ax^2 + bx + c$ into vertex form $y = a(x - h)^2 + q$. This is best illustrated with an example:

Example

Write $y = -2x^2 + 12x - 22$ in vertex form.

Step 1: Rewrite it in the standard form.

Factor out –2 from x^2 and x term: $y = -2(x^2 - 6x) - 22$

Step 2: Add and subtract the square of half the number multiplying the x term:

$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

The equation becomes $y = -2(x^2 - 6x + 9 - 9) - 22$

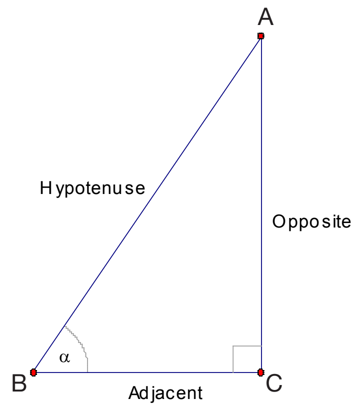
Step 3: Move –9 from the parenthesis:
$y = -2(x^2 - 6x + 9) + (-2)(-9) - 22$

Step 4: Write $(x^2 - 6x + 9)$ in complete square form:
$y = -2(x - 3)^2 + 18 - 22$

Simplify the constants: $y = -2(x - 3)^2 - 4$

The vertex of the parabola is (3, –4).

Trigonometric Ratios

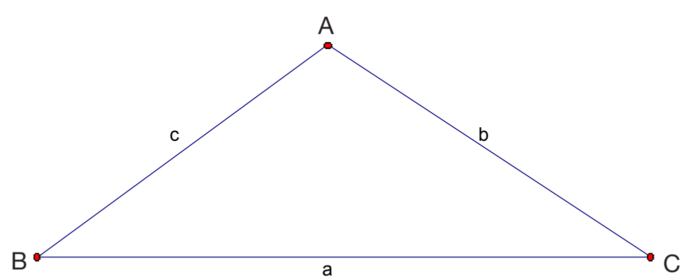


$$\sin(\alpha) = \frac{opposite}{hypotenuse}$$

$$\cos(\alpha) = \frac{adjacent}{hypotenuse}$$

$$\tan(\alpha) = \frac{opposite}{adjacent}$$

Sine Law and Cosine Law



Upper-case letters represent the angle at the vertex and lower-case letters the length of the opposite side.

Cosine Law

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

$$b^2 = a^2 + c^2 - 2ac\cos(B)$$

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Sine Law

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Factor Theorem

If $f(x)$ is a polynomial of degree 3 or higher and $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

Example

Factor $f(x) = x^3 - 7x + 6$.

Solution

Substitute values of x that are factors of the last term, which is 6 until you obtain a value of 0.

$f(1) = 1 - 7 + 6 = 0 \quad \therefore\ x - 1$ is a factor of $f(x)$.

$$
\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
x - 1 & x^3 \quad - 7x + 6 \\
 & \underline{x^3 - x^2} \\
 & \quad x^2 - 7x \\
 & \quad \underline{x^2 - \ \ x} \\
 & \qquad -6x + 6 \\
 & \qquad \underline{-6x + 6} \\
 & \qquad\qquad 0
\end{array}
$$

$\therefore\ f(x) = (x - 1)(x^2 + x - 6)$

$\quad\quad\ \ = (x - 1)(x + 3)(x - 2)$

(**Note**: If you are familiar with synthetic division, it is acceptable in this course.)

MCV4U-A

1

Rates of Change

Introduction

Did you know that the history of calculus can be traced back to 1800 BCE? Today, calculus can open the door to a variety of careers. Many jobs deal with rates, how they change, or how they are affected by different conditions. Calculus examines these rates of change and is used in many engineering, science, and business occupations.

In this lesson, you will look at the rate of change and explore the differences between the average rate of change over an interval and the instantaneous rate of change at a point. Don't despair—as this lesson progresses, you will learn what all of these terms mean.

You will start by looking at a number of rates-of-change applications, such as population growth and displacement of an object. Graphical and algebraic representations of the applications will be presented. An applet on *ilc.org* will be used to collect data and compare the approximate instantaneous rate of change at a point and the average rate of change over an interval.

Estimated Hours for Completing This Lesson	
Rate of Change	2
Instantaneous Rate of Change at a Point	1
Estimating the Rate of Change at a Given Point	1
Key Questions	1

For this lesson, there is an interactive tutorial on your course page. You may find it helpful during, or at the end of, this lesson to work through the tutorial called "Slope of a Tangent."

What You Will Learn

After completing this lesson, you will be able to

- describe real-world applications of rate of change
- distinguish between the average rate of change and the instantaneous rate of change

Rate of Change

What does the rate of change mean and why is it important to study it? Start with a quick example.

The following data represent the population of cranes around Lake Buckhorn:

Year	Population
1990	160
1991	177
1992	196
1993	217
1994	240
1995	265
1996	292
1997	321
1998	352
1999	385
2000	420
2001	457
2002	496
2003	537
2004	580
2005	625

An important question is how the population changes over time. Say you want to study the change between 1990 and 1995. You can observe that the population grew by 265 – 160 = 105 over 5 years. You want to find out the rate of change of the population per year.

It is calculated as follows:

$$\frac{265 - 160}{1995 - 1990} = \frac{105}{5}$$
$$= 21 \text{ cranes/year}$$

You can conclude that, on average, the population grew by 21 cranes per year between 1990 and 1995.

Can you do the same calculation for the population between 2001 and 2004? What is the average increase per year? Your answer should be an average of 41 cranes per year.

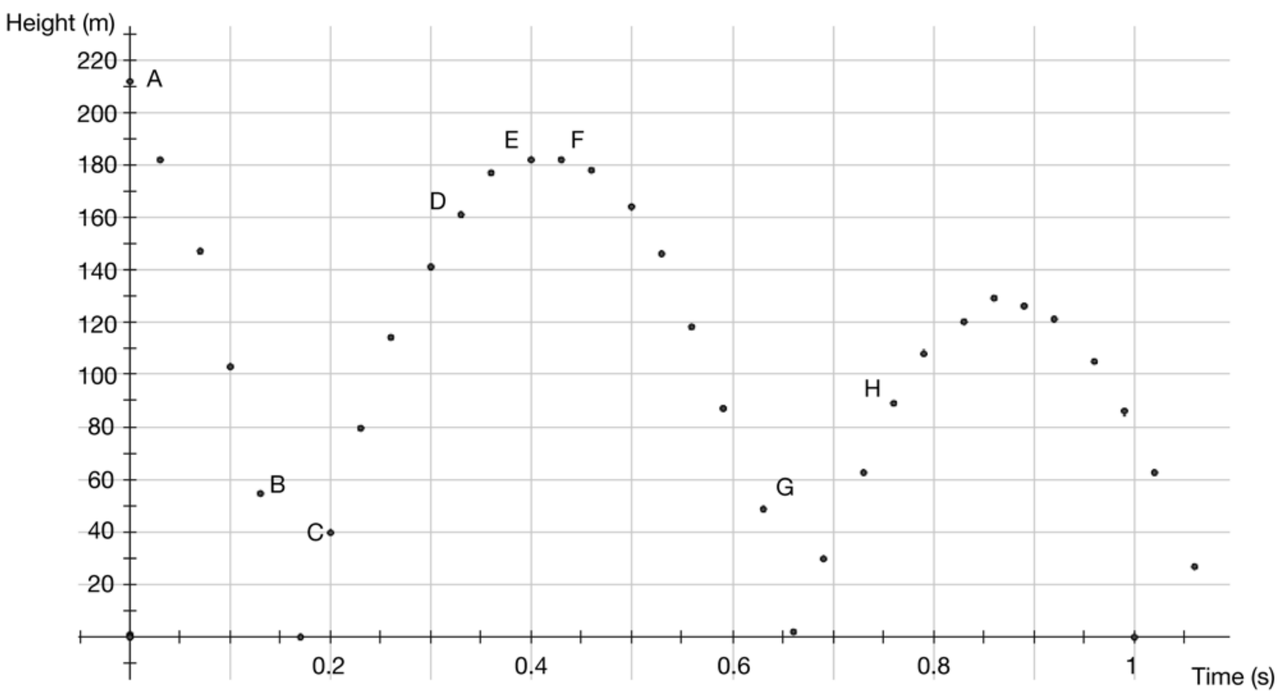
Now you're ready for a formal definition for the rate of change of a function *f*.

> The **average rate of change** of a function with respect to x between a and b is $\dfrac{f(b)-f(a)}{b-a}$.

The rate of change of a function can be applied to a number of real-life scenarios such as average velocity.

Example

Evelyn dropped a ball from the top of a building and recorded the height (in m) at various times (in s). A graph of the data she collected is provided:



Evelyn wants to determine the rate of change of the height at various time intervals; that is, the average velocity of the ball.

Solution

Perform the calculations and ensure that Evelyn's numbers are correct:

Interval	Coordinates of End Points		Average Rate of Change
AB	$A(0, 212)$	$B(0.13, 55)$	$\frac{f(b)-f(a)}{b-a} = \frac{55-212}{0.13-0}$ $= \frac{-157}{0.13}$ ≈ -1207.69 m/s
BC	$B(0.13, 55)$	$C(0.2, 40)$	–214.29 m/s
CD	$C(0.2, 40)$	$D(0.33, 161)$	930.77 m/s
DE	$D(0.33, 161)$	$E(0.4, 182)$	300 m/s
EF	$E(0.4, 182)$	$F(0.43, 182)$	0 m/s
FG	$F(0.43, 182)$	$G(0.63, 49)$	–665 m/s
GH	$G(0.63, 49)$	$H(0.72, 89)$	444.44 m/s

The average rate of change between A and B is negative. What does this tell you? It tells you that on average the height is decreasing between the two points. You will notice on the graph that the height of the ball indeed decreases.

Connection Between the Average Rate of Change and the Slope

Given two points, $A(a, f(a))$ and $B(b, f(b))$, on the graph of a function f, what does the formula of the average rate of change remind you of?

$$\text{Average Rate of Change} = \frac{f(b) - f(a)}{b - a}$$

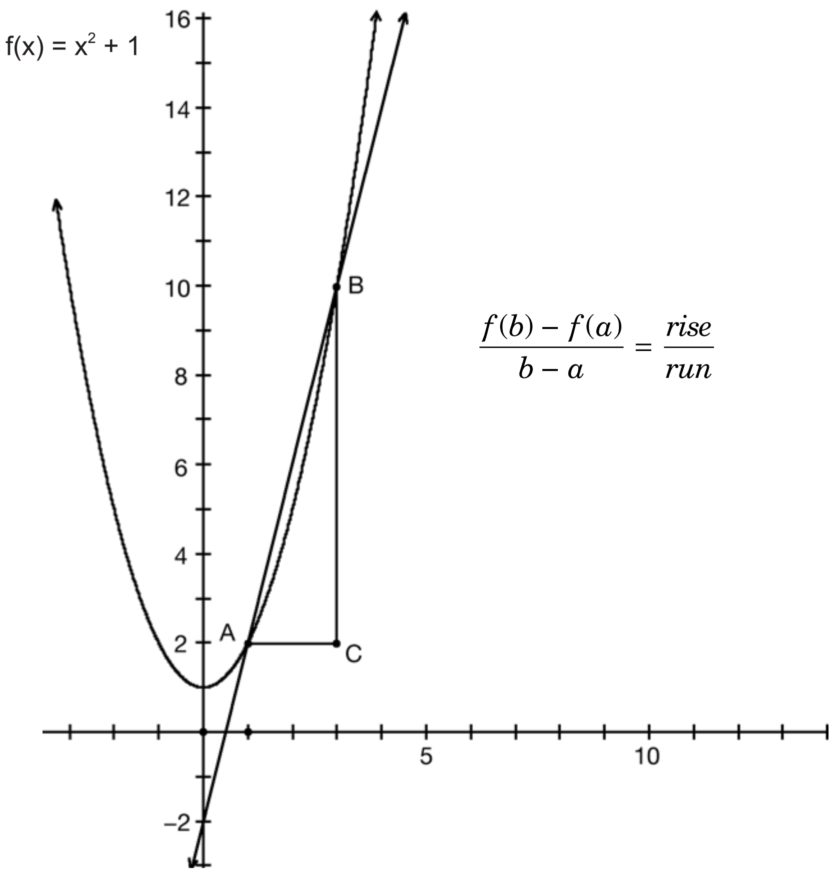
Explore this further with an example.

Example

You are given the function $f(x) = x^2 + 1$ between $x = 1$ and $x = 3$.

$f(1) = (1)^2 + 1 = 2$ and $f(3) = 3^2 + 1 = 10$.

The two points are (1, 2) and (3, 10) and are labelled A and B on the graph.



Do you see a connection between the average rate of change of the function f between 1 and 3 and the line that passes through the points A and B?

Solution

The average rate of change of f over the interval $1 \le x \le 3$ is

$$\begin{aligned}\frac{f(b)-f(a)}{b-a} &= \frac{f(3)-f(1)}{3-1}\\ &= \frac{10-2}{3-1}\\ &= \frac{8}{2}\\ &= 4\end{aligned}$$

The value you calculated is the rise over the run between the points A and B. You can conclude that the average rate of change as x moves from a to b is the slope of the line connecting the points A and B. The line that passes through A and B is called a secant of the function $f(x)$.

> A **secant line** of a curve is a line that intersects the curve through two points.

Apply a real-life scenario to this concept.

Example

A stone is tossed from a bridge 15 m above the water. The height h of the stone, in metres, above the water at t seconds is represented by the following function:

$h(t) = -4.9t^2 + 12t + 15, t > 0$

a) Sketch the graph of the height with respect to t when $0 \le t \le 3$.

b) Find the average velocity of the stone between the time it was thrown and 2 seconds.

c) When does the stone hit the water?

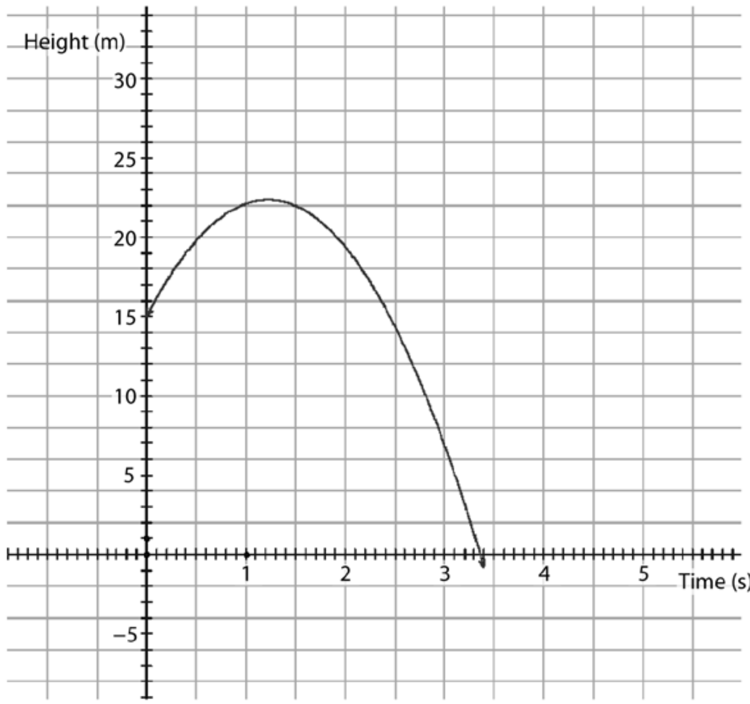
Solution

a) You should recognize the equation as being that of a parabola. Recall that a parabola is the graph of a quadratic function. A quadratic function has the form $y = ax^2 + bx + c$.

Start with a table of values where t varies between 0 and 3 seconds:

t **(s)**	***h*(*t*) (m)**
0	$-4.9(0)^2 + 12(0) + 15 = 15$
1	$-4.9(1)^2 + 12(1) + 15 = 22.1$
2	$-4.9(2)^2 + 12(2) + 15 = 19.4$
3	$-4.9(3)^2 + 12(3) + 15 = 6.9$

Next, graph the points on a grid and draw the parabola that passes through the points.



b) The average velocity between 0 and 2 seconds =

$$\frac{h(2) - h(0)}{2 - 0} = \frac{19.4 - 15}{2}$$
$$= 2.2 \text{ m/s}$$

c) The height of the stone is 0 when it hits the water. Substitute 0 for the height in the equation and solve for t:

$h(t) = -4.9t^2 + 12t + 15$

$-4.9t^2 + 12t + 15 = 0$

Use the quadratic formula to solve:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-12 \pm \sqrt{(12)^2 - 4(-4.9)(15)}}{(2)(-4.9)}$$
$$= \frac{-12 \pm \sqrt{438}}{-9.8}$$
$$= \frac{-12 \pm 20.93}{-9.8}$$
$$t = 3.36 \text{ or } t = -0.91$$

(Since t is time, the negative value is inadmissable and you can ignore it.)

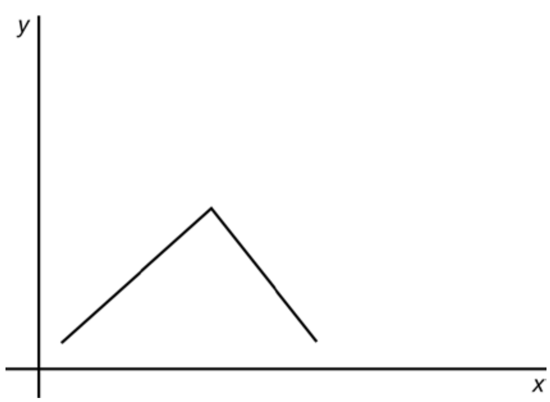
The stone hits the water at $t = 3.36$ seconds. If you check the sketch of the function, this value seems correct because the curve intersects the x-axis at $t = 3.36$ seconds.

Instantaneous Rate of Change at a Point

The speed of a bicycle usually varies, depending on how hard the cyclist is pedalling or if the cyclist is going uphill or downhill. This speed is referred to as the instantaneous speed. This is different from the average speed (average rate of change), which is the distance travelled divided by the time taken to travel that distance. In this case, the average rate of change does not give very accurate information about the behaviour of the cyclist you are studying at a given time t.

Instantaneous rate of change is the measure of the rate of change for a continuous smooth function at a specific point on the function.

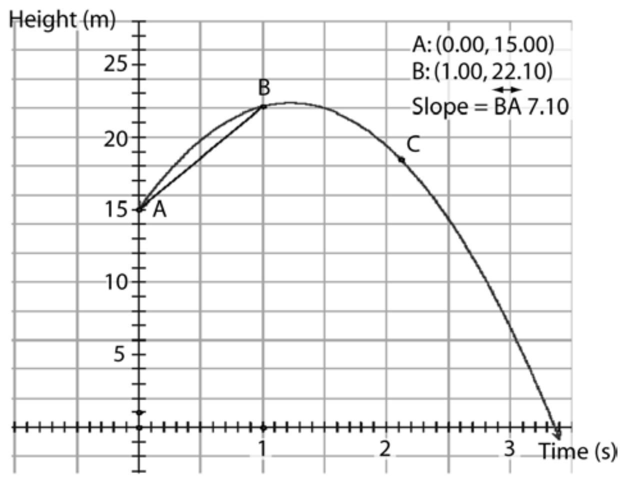
A smooth function is a function with no corners. The following graph is not a smooth function:



The Secant Method to Approximate Instantaneous Rate of Change

In the previous example, you calculated the average velocity of the stone between 0 and 2 seconds. What if you want to find the velocity of the stone exactly when $t = 1$?

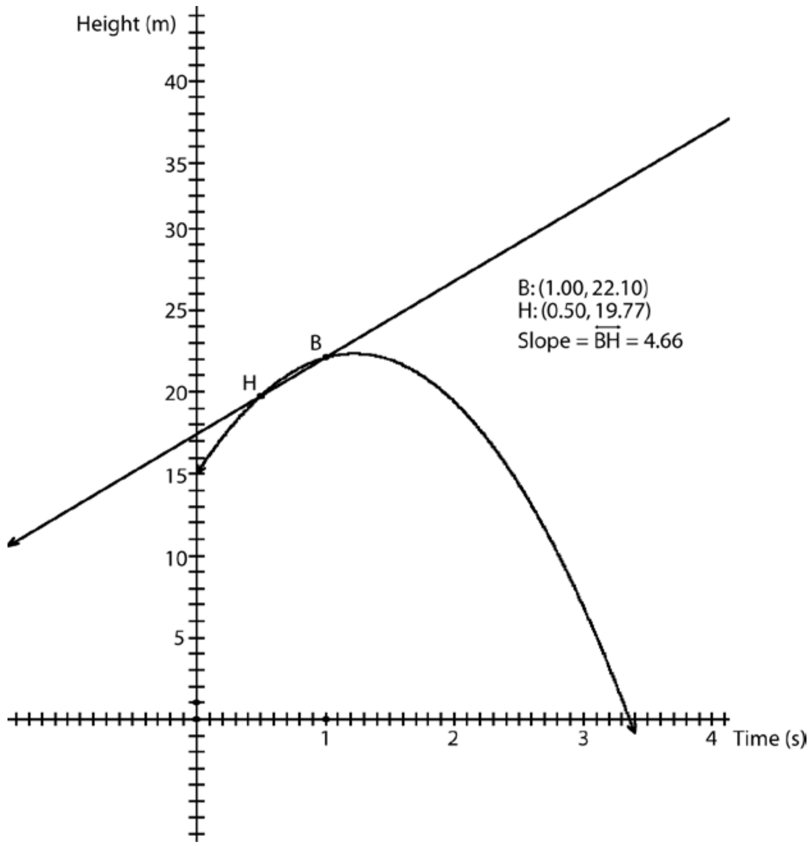
You can start by calculating the average velocity between $t = 0$ and $t = 1$. The slope of AB gives the average rate of change of the height between $t = 0$ and $t = 1$.



The average rate of change between $t = 0$ and $t = 1$:

$$\frac{h(1) - h(0)}{1} = \frac{22.1 - 15}{1}$$
$$= 7.1 \text{ m/s}$$

To get a better estimate, calculate the average velocity between 0.5 and 1.



Calculating the average rate of change between 0.5 and 1 gives you a better approximation of the velocity at $t = 1$ second. You do this by calculating the slope of the secant HB.

$$\frac{h(1) - h(0.5)}{1 - 0.5} = \frac{22.1 - 19.77}{0.5}$$

The answer you get is 4.66 m/s.

You can repeat this process for smaller intervals if you want to get an even better estimate of the rate of change at $t = 1$. The closer the second point of the secant is to B, the better the estimate.

Support Questions

(do not send in for evaluation)

1. The population of flies in an experiment is given by the following equation: $N(t) = t^2 + 4t + 1$, where t is time in seconds and $0 \le t \le 10$.

   a) Draw the graph of the population with respect to time.

   b) Find the average rate of change between 1 and 10 seconds.

   c) Use the secant method to approximate the instantaneous rate of change at $t = 5$ seconds.

2. The height of a ball thrown in the air in metres after t seconds is given by $h(t) = -4.9t^2 + 30t$.

   a) Draw a graph of the height with respect to time.

   b) Find the average velocity for the first 2 seconds.

   c) When does the ball hit the ground?

   d) Use the secant method to approximate the instantaneous rate of change at $t = 1.5$ seconds.

There are Suggested Answers to Support Questions at the end of this unit.

Estimating the Rate of Change at a Given Point

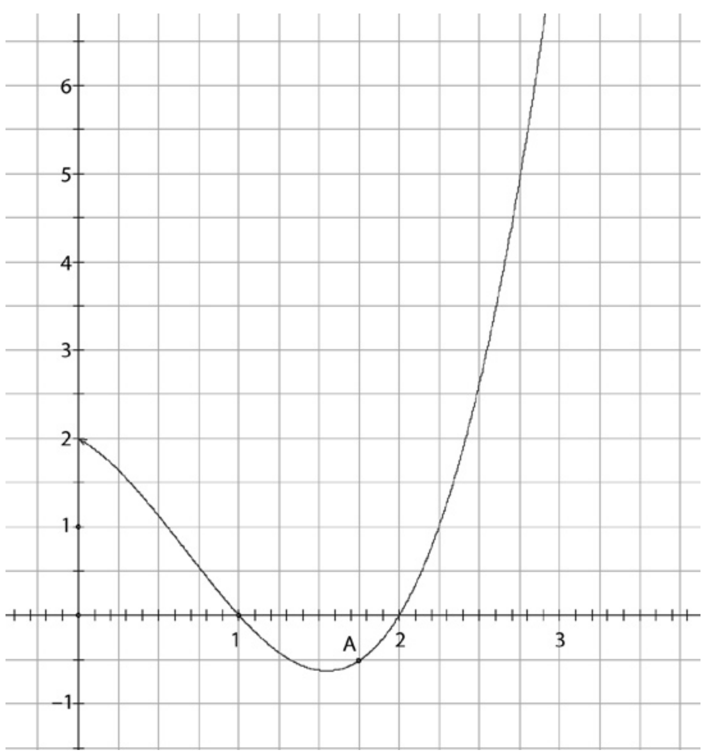
Go to the ILC website and launch the applet "Lesson 1 Activity 1."

As you drag point A along the curve, the secant connecting A to B moves and the slope changes accordingly. The closer A is to B, the better the estimate of the rate of change at $t = 1$. You will notice that the secant looks like a tangent to the curve when A is at B and the slope of the line is 2.2 m/s.

Recall that a tangent to a curve is a line that makes contact with a curve at one point. You can observe that the rate of change at $t = 1$ is approximately 2.2 m/s.

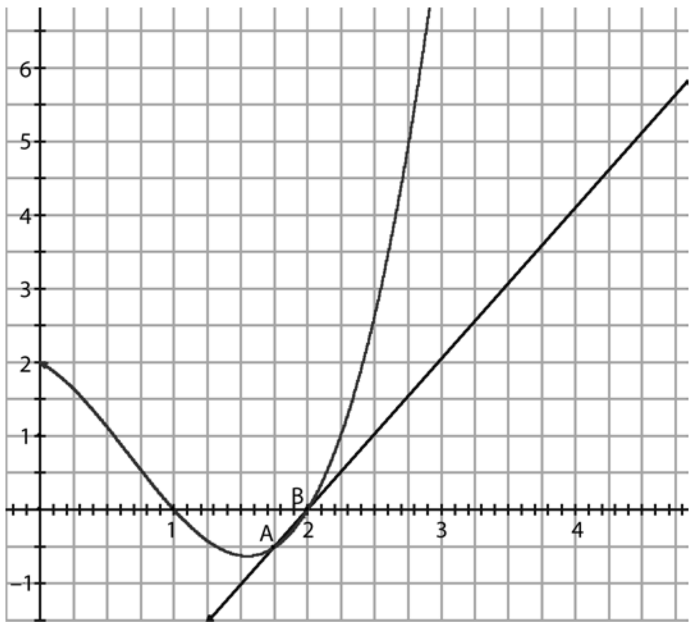
Example

For the following graph, find an approximate value for the slope of the tangent at point A by using the secant method. What is the instantaneous rate of change at point A?



Solution

Approximate the slope of the tangent with the slope of the secant on the curve between $x = 1.75$ and $x = 2$.



On the graph of the function, the coordinates are approximately $A(1.75, -0.5)$ and $B(2, 0)$.

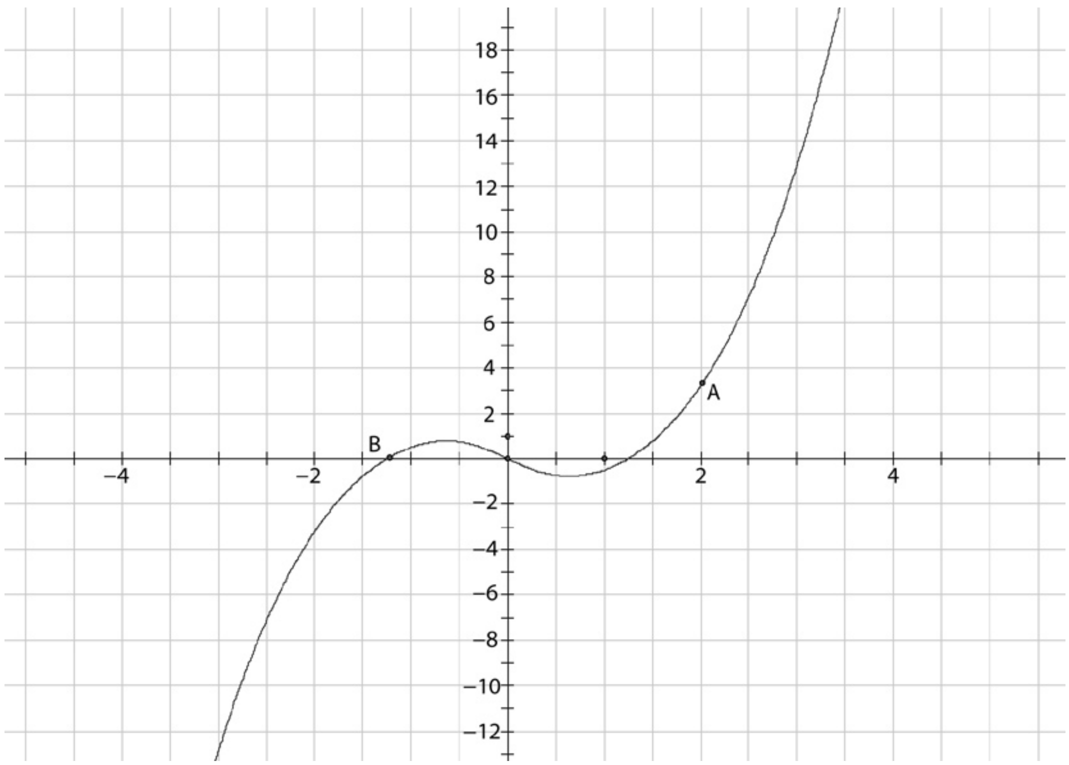
$$\begin{aligned}\text{Slope of } AB &= \frac{0-(-0.5)}{2-1.75}\\ &= \frac{0.5}{0.25}\\ &= 2\end{aligned}$$

Using the secant method, you can conclude that the slope of the tangent at point A is approximately 2, hence the instantaneous rate of change is approximately 2.

Support Questions

(do not send in for evaluation)

3. Find an approximate value for the slope of the tangent at point A by using the secant method:



4. As a bicycle moves from a stop sign, the distance in metres of the bicycle from the stop sign after t seconds is given by $d(t) = 0.5t^2$ when $2 \leq t \leq 12$.

   a) Using the Graphing Applet on your course page at *ilc.org*, sketch the distance with respect to time on the interval $2 \leq t \leq 12$.

   b) What is the average velocity when $2 \leq t \leq 6$?

   c) Approximate the instantaneous rate of change when $t = 3$.

   d) Approximate the instantaneous rate of change when $t = 4$.

   e) When is the cyclist going faster, at $t = 3$ or $t = 4$?

   f) Is the bicycle slowing down or going faster as its distance from the stop sign increases?

Conclusion

In this lesson, you approximated the rate of change by approximating the slope of the tangent to a curve at a specific point. In Lesson 2, you will learn about limits. Limits give a formal way to talk about the value of a function as x approaches a specific value. This will help you find an exact value of the instantaneous rate of change at a point.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(19 marks)

1. The height in metres of a ball dropped from the top of the CN Tower is given by $h(t) = -4.9t^2 + 450$, where t is time elapsed in seconds.

   a) Draw the graph of h with respect to time. **(3 marks)**

   b) Find the average velocity for the first 2 seconds after the ball was dropped. **(1 mark)**

   c) Find the average velocity for the following time intervals: **(3 marks)**

   i) $1 \le t \le 4$

   ii) $1 \le t \le 2$

   iii) $1 \le t \le 1.5$

   d) Use the secant method to approximate the instantaneous velocity at $t = 1$ second. **(1 mark)**

2. The mass M in grams of undissolved sugar left in a teacup after t seconds is given by $M = 10.5 - 0.4t^2$.

   a) When will all the sugar dissolve? **(2 marks)**

   b) Find the average rate of change in the interval $0 \leq t \leq 1$. **(1 mark)**

   c) Draw on graphing paper a graph of M with respect to t and use the secant method to approximate the instantaneous rate of change at $t = 2$ seconds. **(3 marks)**

3. As a car moves from a traffic light, the distance in metres of the car from the traffic light after t seconds is given by $d(t) = 2t^2$.

   a) Draw a graph on graphing paper of the distance versus time. **(3 marks)**

   b) Calculate the average speed when $4 \leq t \leq 7$. **(1 mark)**

   c) Use the secant method to approximate the instantaneous velocity at $t = 4$ seconds. **(1 mark)**

Now go on to Lesson 2. Do not submit your coursework to ILC until you have completed Unit 1 (Lessons 1 to 5).

