YORK UNIVERSITY

Faculty of Arts

Atkinson Faculty of Liberal and Professional Studies

AS/AK/MATH 1550 6.0 B

In-class Exam #2

Solutions

1. (a) (6 points) Let $A = \begin{bmatrix} 1 & 6 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Find A^{-1} and use it to solve the system of linear equations $AX = B$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix}$.

No points will be given for any other solution.

Answer:

Using the method of reduction,

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \overset{(R_1+(-6)R_2)}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -6 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{(\frac{1}{2}R_3)}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -6 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \overset{(R_1+(1)R_3)}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & 1/2 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right]$$

$$\overset{(R_2+(-1)R_3)}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right].$$

So, $A^{-1} = \begin{bmatrix} 1 & -6 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$.

Hence, $X = A^{-1}B = \begin{bmatrix} 1 & -6 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ -2 \end{bmatrix}$.

(b) (3 points) Write the matrix A^{-1} from part (a) as a product of elementary matrices.

Answer:

Applying the EROs from part (a) to the identity matrix I_3, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{(R_1+(-6)R_2)}{\longrightarrow} \begin{bmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1,$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{(\frac{1}{2}R_3)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} = E_2,$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(R_1+(1)R_3)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3,$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(R_2+(-1)R_3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = E_4.$$

Therefore,

$$E_4 \cdot E_3 \cdot E_2 \cdot E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -6 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} = A^{-1}.$$

2. (a) (2 points) Let A be a square matrix of order 3, with $det(A) = 5$. Evaluate $det([3A^{-1}]^T)$.

   Answer:

   $det([3A^{-1}]^T) = det(3A^{-1}) = 3^3 det(A^{-1}) = 3^3 \dfrac{1}{det(A)} = \dfrac{27}{5}.$

   (b) (4 points) If $\begin{vmatrix} 2 & k & 3 & 0 \\ k & 4 & 1 & 1 \\ 0 & -1 & 5 & 1 \\ 0 & 1 & 4 & 2 \end{vmatrix} = 36$, find the value(s) of k.

   Answer:

   Expanding the determinant along the first column,

   $$\begin{vmatrix} 2 & k & 3 & 0 \\ k & 4 & 1 & 1 \\ 0 & -1 & 5 & 1 \\ 0 & 1 & 4 & 2 \end{vmatrix} = (-1)^{1+1}2 \begin{vmatrix} 4 & 1 & 1 \\ -1 & 5 & 1 \\ 1 & 4 & 2 \end{vmatrix} + (-1)^{1+2}k \begin{vmatrix} k & 3 & 0 \\ -1 & 5 & 1 \\ 1 & 4 & 2 \end{vmatrix}$$

   $= 2 \cdot 18 - k(k \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} + (-1)^{1+2}3 \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix})$

   $= 36 - k[6k - 3(-3)] = 36 - 6k^2 - 9k.$

   So, $36 - 6k^2 - 9k = 36 \implies 6k^2 + 9k = 0$ or $k(6k + 9) = 0.$

   Hence, $k = 0$ or $k = -\dfrac{3}{2}.$

   (c) (3 points) If $\begin{vmatrix} a & 3 & c \\ k & 1 & m \\ p & 2 & r \end{vmatrix} = 8$, and the system $\begin{cases} ax_1 + bx_2 + cx_3 = 3 \\ kx_1 + lx_2 + mx_3 = 1 \\ px_1 + qx_2 + rx_3 = 2 \end{cases}$

   has a unique solution in which $x_2 = 2$, then $\begin{vmatrix} a & b & c \\ k & l & m \\ p & q & r \end{vmatrix} =$ ______________

   Answer:

   By the Cramer's rule,

   $x_2 = \dfrac{\Delta_2}{\Delta}$, where $\Delta_2 = 8.$

So, $2 = \dfrac{8}{\Delta}$, and $\Delta = \dfrac{8}{2} = 4$.

3. (a) (5 points) Solve the following equations for x :

   i. $7^{4\log_7 x} = 81$

   Answer:
   $7^{4\log_7 x} = 3^4$,
   $x^4 = 3^4$,
   $x = \pm 3$.
   But, log is not defined for $x < 0$.
   Hence, $x = 3$.

   ii. $\log_3 9 + \log_3 8 - \log_3 2x = \log_6 6$.

   Answer:
   $\log_3 9 + \log_3 8 - \log_3 2x = 1$,
   $\log_3 3^2 + \log_3(\dfrac{8}{2x}) = 1$,
   $2 + \log_3(\dfrac{4}{x}) = 1$,
   $\log_3(\dfrac{4}{x}) = -1$,
   $\dfrac{4}{x} = 3^{-1}$,
   $x = 4 \cdot 3 = 12$.

   (b) (3 points) At what nominal rate r per annum compounded continuously will your investment of \$2 000 increase 50% in value in 3 years. (Answer correct to four decimal places.)

   Answer:
   50% of \$2 000 is \$1 000.
   We have, $S = Pe^{rt}$. Substituting the values $S = 2\,000 + 1\,000 = 3\,000$, $P = 2\,000$, and $t = 3$(years), we obtain
   $3\,000 = 2\,000e^{r(3)}$,
   $3r = \ln(1.5)$,
   $r = \dfrac{\ln(1.5)}{3} \approx 0.1352$ or 13.52%.

4. Find each of the following limits if it exists:

   (a) (2 points) $\lim_{x\to\infty} \dfrac{3x^2 - 1}{(2x+3)^2}$

   Answer:
   $$\lim_{x\to\infty} \frac{3x^2 - 1}{(2x+3)^2} = \lim_{x\to\infty} \frac{3x^2 - 1}{4x^2 + 6x + 9} = \lim_{x\to\infty} \frac{3 - \frac{1}{x^2}}{4 + \frac{6}{x} + \frac{9}{x^2}} = \frac{3}{4}.$$

   (b) (2 points) $\lim_{x\to 1} \dfrac{x^2 + x - 2}{x^2 - 4x + 3}$

   Answer:
   $$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - 4x + 3} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x-3)} = \lim_{x\to 1} \frac{x+2}{x-3} = -\frac{3}{2}.$$

(c) (2 points) $\lim_{x\to 0}(x-\frac{5}{x})$

Answer:

$\lim_{x\to 0^-}(x-\frac{5}{x})=\infty$ and $\lim_{x\to 0^+}(x-\frac{5}{x})=-\infty$.

So, the limit $\lim_{x\to 0}(x-\frac{5}{x})$ does not exist.

(d) (2 points) $\lim_{x\to 2^-}\frac{\mid x-2\mid}{x^2-2x}$

Answer:

$\lim_{x\to 2^-}\frac{\mid x-2\mid}{x^2-2x}=\lim_{x\to 2^-}\frac{2-x}{x^2-2x}=-\lim_{x\to 2^-}\frac{x-2}{x(x-2)}=-\lim_{x\to 2^-}\frac{1}{x}=-\frac{1}{2}.$

5. (a) (4 points) Use the definition of the derivative to find $f'(x)$ if $f(x)=\sqrt{3x-2}$.
No points will be given for any other solution.

Answer:

$f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\sqrt{3x-2}-\sqrt{3a-2}}{x-a}$

$=\lim_{x\to a}\frac{(\sqrt{3x-2}-\sqrt{3a-2})(\sqrt{3x-2}+\sqrt{3a-2})}{(x-a)(\sqrt{3x-2}+\sqrt{3a-2})}$

$=\lim_{x\to a}\frac{(3x-2)-(3a-2)}{(x-a)(\sqrt{3x-2}+\sqrt{3a-2})}=\lim_{x\to a}\frac{3(x-a)}{(x-a)(\sqrt{3x-2}+\sqrt{3a-2})}$

$=\lim_{x\to a}\frac{3}{\sqrt{3x-2}+\sqrt{3a-2}}=\frac{3}{\sqrt{3a-2}+\sqrt{3a-2}}=\frac{3}{2\sqrt{3a-2}}.$

(b) (2 points) Find the equation of the tangent line to the graph of the function $y=\sqrt{3x-2}$ at the point $(2,2)$.

Answer:

From part (a) $f'(x)=\frac{3}{2\sqrt{3x-2}}$.

So, the slope of the tangent line to the graph of the function $y=\sqrt{3x-2}$ at the point $(2,2)$ is $f'(2)=\frac{3}{2\sqrt{3\cdot 2-2}}=\frac{3}{4}$.

Hence, the slope-intercept equation of the tangent line will be $y=\frac{3}{4}x+b$. Substituting $x=2$ and $y=2$, we find $b=2-\frac{6}{4}=\frac{1}{2}$, and the equation will be $y=\frac{3}{4}x+\frac{1}{2}$.

(c) (2 points) Let $g(x)=\begin{cases}1-x^2 & \text{if } x<1\\ x+1 & \text{if } x\geq 1\end{cases}$

Then

i. $\lim_{x\to 1^-}g(x)=0$,

ii. $\lim_{x\to 1^+}g(x)=2$,

iii. $\lim_{x \to 1} g(x) =$ (does not exist).

iv. Is $g(x)$ continuous at $x = 1$? No.

6. (a) (2 points) If $f(2) = 5,\ g(2) = -5,\ f'(2) = 1.2,\ g'(2) = 0.8$, then
$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$,
$(fg)'(2) = f(2)g'(2) + f'(2)g(2) = 5(0.8) + (1.2)(-5) = -2.$

(b) (3 points) The total cost c of producing q widgets ($q > 5$) is estimated by $c = 24 + 4q + 20q^{-1}$. Find the marginal cost when 15 widgets are produced.

Answer:

$$\frac{dc}{dq} = 4 + (-1)\frac{20}{q^2} = 4 - \frac{20}{q^2}.$$

$$\frac{dc}{dq}|_{q=15} = 4 - \frac{20}{(15)^2} = \$3.91.$$

(c) (3 points) If $p = \dfrac{q+14}{q+4}$ is a demand equation, find the rate of change of the price with respect to quantity q when $q = 10$.

Answer:

$$\frac{dp}{dq} = \frac{d}{dq}\left(\frac{q+14}{q+4}\right) = \frac{(q+4) - (q+14)}{(q+4)^2} = -\frac{10}{(q+4)^2}.$$

$$\frac{dp}{dq}|_{q=10} = -\frac{10}{(10+4)^2} = -\frac{5}{98}.$$

The end