Economics 1540

Introductory
Mathematics for Economists II

Lecture 1

1. Functions of Several Variables

(a) Functions of two variables
(b) Partial derivatives with two variables
(c) Geometric representation
(d) Surfaces and distance
(e) Functions of more variables
(f) Partial Derivatives with more variables
(g) Economic Applications
(h) Partial Elasticities

- Economics 1530 examined functions of one variable.
- Economics is generally about choices between two or more alternatives.
- We typically model these as functions of two or more variables.
- Most of what was examined in 1530 easily generalizes to more than one variable.
- We now examine functions of two variables.

(a) Functions of two Variables

- A function of two variables, x and y, with domain D, is a rule that assigns a specified number, $f(x, y)$, to each point, (x, y), in D.
- Let $z = f(x, y)$
- z is usually called the "dependent" variable.
- x and y are called the "independent" variables or the "arguments" of $f(x, y)$.

Example

$$f(x, y) = 2x + x^2y^3, \qquad (x, y) \in \Re^2 \quad (1)$$

- Let's evaluate f for a few values of (x, y)
- E.g., $(1, 0)$, $(0, 1)$, $(-2, 3)$ and $(a + 1, b)$

$$f(1, 0) = 2(1) + 1^2(0^3) = 2 \qquad (2)$$

$$f(0, 1) = 2(0) + 0^2(1^3) = 0 \qquad (3)$$

$$\begin{aligned} f(-2,3) &= 2(-2) + (-2)^2(3^3) \\ &= -4 + 4(27) \\ &= 104 \end{aligned} \tag{4}$$

$$f(a+1,b) = 2(a+1) + (a+1)^2(b^3) \tag{5}$$

Example

- Frisch and Haavelmo estimated the demand for milk.

$$x = \frac{Ar^{2.08}}{p^{1.5}} \tag{6}$$

- x represents milk consumption, r represents income per family and p represents price.

Example

$$F(S, E) = 2.26 S^{0.44} E^{0.48} \tag{7}$$

- S represents stock of lobsters, E represents harvest/catch.
- Let's evaluate F at (tS, tE).

$$\begin{aligned} F(tS, tE) &= 2.26 (tS)^{0.44} (tE)^{0.48} \\ &= 2.26 t^{0.92} S^{0.44} E^{0.48} \\ &= t^{0.92} 2.26 S^{0.44} E^{0.48} \\ &= t^{0.92} F(S, E) \end{aligned} \tag{8}$$

- This is an example of a homogeneous function.
- It is homogeneous of degree 0.92
- When the inputs are increased by the factor t the output was increased by a factor of $t^{0.92}$
- for example if we double inputs the output is less than doubled.

Example

- The previous example is a special case of a Cobb Douglas function:

$$F(x, y) = Ax^a y^b \tag{9}$$

- A, a, b are constants and $x, y > 0$.

- Let's evaluate F at $(2x, 2y)$

$$\begin{aligned} F(2x, 2y) &= A(2x)^a(2y)^b \\ &= 2^{a+b}Ax^ay^b \\ &= 2^{a+b}F(x, y) \end{aligned} \tag{10}$$

- This is also an example of a homogeneous function.
- It is homogeneous of degree $a + b$.

(b) Partial derivatives with two variables

- In the single variable case with $y = f(x)$ we defined the derivative as

$$\frac{dy}{dx} = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} \tag{11}$$

- this is interpreted as the instantaneous rate of change of y as we change x.

- Now, with two variables we want to know how the dependent variable changes as one of the independent variable changes, holding the other constant

Example

- if $z = x^3 + 2y^2$ consider holding y constant and changing x:

$$\left.\frac{dz}{dx}\right|_{y \text{ constant}} = 3x^2 \tag{12}$$

- Similarly, holding x constant we have

$$\left.\frac{dz}{dy}\right|_{x \text{ constant}} = 4y \tag{13}$$

- The usual notation is:

$$\frac{\partial z}{\partial x} = \left.\frac{dz}{dx}\right|_{y \text{ constant}} \tag{14}$$

$$\frac{\partial z}{\partial y} = \left.\frac{dz}{dy}\right|_{x \text{ constant}} \tag{15}$$

where the ∂ is read as "partial."

- Formally, we define:

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= \lim_{\delta\to 0}\frac{f(x+\delta,y)-f(x,y)}{\delta}\\ \frac{\partial f(x,y)}{\partial y} &= \lim_{\delta\to 0}\frac{f(x,y+\delta)-f(x,y)}{\delta}\end{aligned} \tag{16}$$

Definition:

- Suppose $z = f(x, y)$. The partial derivative of z or $f(x, y)$ with respect to x, is the derivative of $f(x, y)$ with respect to x when y is held constant.
- The partial derivative of z or $f(x, y)$ with respect to y is the derivative of $f(x, y)$ with respect to y with x is held constant.
- These are often written as $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, respectively.
- This is very easy. It is just like differentiation in the single variable case except that the variables besides the one of interest are treated as constants.

Example:

$$f(x,y) = x^3y + x^2y^2 + x + y^2 \tag{17}$$

$$\frac{\partial f}{\partial x} = 3x^2y + 2xy^2 + 1 \tag{18}$$

$$\frac{\partial f}{\partial y} = x^3 + 2x^2y + 2y \tag{19}$$

Example:

$$f(x,y) = \frac{xy}{x^2+y^2} \tag{20}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} \\
&= \frac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2} \\
&= \frac{y^3 - x^2y}{(x^2+y^2)^2}
\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{(x^2+y^2)x - xy2y}{(x^2+y^2)^2} \\ &= \frac{x^3 - y^2x}{(x^2+y^2)^2}\end{aligned} \tag{22}$$

Example:

$$x = Ap^{-1.5}r^{1.08}, \qquad A, p, r > 0 \tag{23}$$

$$\frac{\partial x}{\partial p} = -1.5Ap^{-2.5}r^{1.08} < 0 \tag{24}$$

$$\frac{\partial x}{\partial r} = 1.08Ap^{-1.5}r^{0.08} > 0 \tag{25}$$

- In this case, a unit increase in p will unambiguously lead to a decrease in x; similarly a unit increase in r will unambiguously lead to an increase in x.

- Other notations are quite common

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = z'_x = f'_x(x,y) = f'_1(x,y) = \frac{\partial f(x,y)}{\partial x} \tag{26}$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = z'_y = f'_y(x,y) = f'_2(x,y) = \frac{\partial f(x,y)}{\partial y} \tag{27}$$

Approximations to Partial Derivatives:

- Recall that

$$\frac{df(x)}{dx} \simeq f(x+1) - f(x) \tag{28}$$

- Similarly,

$$\frac{\partial f(x,y)}{\partial x} \simeq [f(x+1,y) - f(x,y)] \tag{29}$$

$$\frac{\partial f(x,y)}{\partial y} \simeq [f(x,y+1) - f(x,y)] \tag{30}$$

- The approximations must be used with caution.
- They will be fairly accurate provided that the partial derivatives do not vary too much over the interval of interest.

Higher order partial derivatives:

- With functions of one variable we had higher order derivatives which are derivatives of derivatives.
- We also have second order partial derivatives.

- For example, with $f(x, y)$ we define the following:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \tag{31}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \tag{32}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \tag{33}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \tag{34}$$

Example

$$f(x,y) = x^3y + x^2y^2 + x + y^2 \tag{35}$$

$$\frac{\partial f}{\partial x} = 3x^2y + 2xy^2 + 1 \tag{36}$$

$$\frac{\partial f}{\partial y} = x^3 + 2x^2y + 2y \tag{37}$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy + 2y^2 \tag{38}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 4xy \tag{39}$$

$$\frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 4xy \tag{40}$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^2 + 2 \tag{41}$$

- We can define still higher order partials e.g., 3^{rd} order and higher

Example

- If

$$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 4xy \tag{42}$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\frac{\partial^2 f}{\partial x \partial y} = 6x + 4y \tag{43}$$

Example:

$$f(x, y) = x^3 e^{y^2} \tag{44}$$

- Find the first and second order partials at $(x, y) = (1, 0)$.

$$\frac{\partial f}{\partial x} = 3x^2 e^{y^2} \tag{45}$$

- Therefore

$$\frac{\partial f(1, 0)}{\partial x} = 3(1)^2 e^0 = 3 \tag{46}$$

- Similarly,

$$\frac{\partial f}{\partial y} = x^3 e^{y^2} 2y = 2x^3 y e^{y^2} \tag{47}$$

$$\frac{\partial f(1,0)}{\partial y} = 2(1)^3(0)e^0 = 0 \tag{48}$$

$$\frac{\partial^2 f}{\partial x^2} = 6xe^{y^2} \tag{49}$$

$$\frac{\partial^2 f(1,0)}{\partial x^2} = 6 \tag{50}$$

$$\frac{\partial^2 f}{\partial y \partial x} = 3x^2 e^{y^2} 2y = 6x^2 y e^{y^2} \tag{51}$$

$$\frac{\partial^2 f(1,0)}{\partial y \partial x} = 0 \tag{52}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6x^2 y e^{y^2} \tag{53}$$

$$\frac{\partial^2 f(1,0)}{\partial x \partial y} = 0 \tag{54}$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^3(e^{y^2} + y2ye^{y^2}) = 2x^3e^{y^2} + \overset{\text{4 not 6}}{6x^3y^2}e^{y^2} \tag{55}$$

$$\frac{\partial^2 f(1,0)}{\partial y^2} = 2 \tag{56}$$