SOLUTION & MARKING SCHEME

Surname	Given Name	Student Number

MATH1013M 3.0 Test no. 2 W2014
Time: 45 minutes Value: 100 marks
No calculators or written material allowed.

(Marks)
(25) Q1.

(10) a) If A, B and C ($C\neq 0$) are constants, evaluate $\lim_{t\to 0} \dfrac{A\,t + \tan(B\,t)}{\sin(C\,t)}$

A correct answer without detailed intermediate work will be given little merit.

$$\lim_{t\to 0}(\ldots) = \lim_{t\to 0}\frac{At}{\sin Ct} + \lim_{t\to 0}\frac{\tan Bt}{\sin Ct}$$

$$= \lim_{t\to 0}\frac{At}{\sin Ct}\cdot\frac{C}{C} + \lim_{t\to 0}\frac{\sin Bt}{\cos Bt}\cdot\frac{1}{\sin Ct}$$

$$= \lim_{t\to 0}\frac{A}{C}\cdot\frac{Ct}{\sin Ct} + \lim_{t\to 0}\frac{1}{\cos Bt}\cdot\lim_{t\to 0}\frac{\sin Bt}{\sin Ct}$$

$$= \frac{A}{C}\lim_{t\to 0}\frac{1}{\frac{\sin Ct}{Ct}} + \lim_{t\to 0}\frac{1}{\cos Bt}\cdot\lim_{t\to 0}\frac{\sin Bt}{\sin Ct}\cdot\frac{Bt}{Bt}\cdot\frac{Ct}{Ct}$$

$$= \frac{A}{C}\cdot\frac{1}{\lim_{t\to 0}\frac{\sin Ct}{Ct}} + \lim_{t\to 0}\frac{1}{\cos Bt}\cdot\lim_{t\to 0}\frac{\sin Bt}{Bt}\cdot\lim_{t\to 0}\frac{1}{\frac{\sin Ct}{Ct}}\cdot\frac{B\cancel{t}}{C\cancel{t}}$$

$$= \frac{A}{C}\cdot 1 + 1\cdot 1\cdot 1\cdot\frac{B}{C}$$

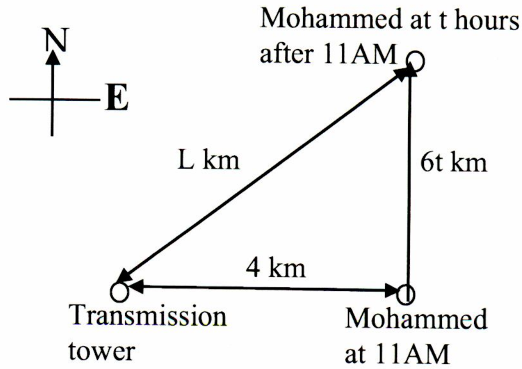
$$= \boxed{\frac{A+B}{C}}$$

TA: look for correct intermediate work and give appropriate partial credit

Full credit require correct intermediate work to be shown

...to be continued...

(15) b) At 11am Mohammed is 4km due east of a cell phone transmission tower. He's jogging due north at 6 km/hour. Refer to the adjacent diagram.
Find an expression for the distance, L km, between Mohammed and the transmission tower t hours after 11am.
Then evaluate the rate at which L is changing, i.e. dL/dt, at 11:30am. Use appropriate units.
Is the distance increasing or decreasing?



Pythagorus Theorem $\Rightarrow$ $L^2 = 4^2 + (6t)^2$ 5

$\therefore \boxed{L = \sqrt{16 + 36t^2}}$ where t is time in hours after 11AM.

Diff. with respect to t :

$$2L \cdot \frac{dL}{dt} = \frac{d}{dt}(16) + \frac{d}{dt}(36t^2)$$

$$= 0 + 2 \cdot 36 \cdot t$$

$$\therefore \boxed{\frac{dL}{dt} = \frac{36t}{L}}$$ 5

at 11:30 am, $t = \frac{1}{2}$; $L = \sqrt{16 + 36 \cdot \left(\frac{1}{2}\right)^2} = \sqrt{16+9} = \sqrt{25} = 5$

NB. (3,4,5) pythagorean triangle

$$\therefore \left.\frac{dL}{dt}\right|_{\substack{(11:30\ AM)\\ \text{or } t=\frac{1}{2}}} = \frac{36 \cdot \frac{1}{2}}{5} = \frac{18}{5} \text{ (or } 3.6) \text{ km/hr.}$$

3 marks 1 mark

Since $\frac{dL}{dt} > 0$, distance is $\boxed{\text{increasing}}$ 1 mark.

...to be continued...

(25) **Q2.** [You may use implicit differentiation in this question.]

(15) a) Evaluate $\left.\dfrac{dy}{dx}\right|_{x=3}$ if $\dfrac{3}{x}+\dfrac{1}{y}=2$

(10) b) Obtain dy/dx and simplify as far as possible if $e^{x/y}=5x+y$

a) Differentiate both sides of given eqn w.r.to ~~x~~ to yield:

$$-\frac{3}{x^2}-\frac{1}{y^2}y'=0$$

$$\therefore \quad y'=-\frac{3y^2}{x^2} \quad 5$$

$$\text{at } x=3,\ \frac{3}{3}+\frac{1}{y}=2 \Rightarrow y=1 \quad 5$$

$$\therefore \quad y'\Big|_{x=3}=-3\cdot\frac{1^2}{3^2}=\boxed{-\frac{1}{3}} \quad 5$$

alternative: use "explicit" differentiation

Solve for y in terms of x: $\frac{1}{y}=2-\frac{3}{x}=\frac{2x-3}{x}$

$$\therefore \quad y=\frac{x}{2x-3}$$

$$\therefore \quad y'=\frac{(2x-3).1-x(.2)}{(2x-3)^2}=\frac{-3}{(2x-3)^2}$$

$$\therefore \quad y'\Big|_{x=3}=\frac{-3}{(6-3)^2}=\text{same as above}$$

b) alternative 1 (brute force)

$$e^{x/y}\frac{d}{dx}\left(\frac{x}{y}\right)=5+y'$$

$$e^{x/y}\frac{(y.1-xy')}{y^2}=5+y'$$

$$(5x+y)(y-xy')=(5+y')y^2 \quad \Leftarrow \text{same as}$$

$$5xy+y^2-5x^2y'-xyy'=5y^2+y^2y'$$

$$5xy-4y^2=y'(5x^2+xy+y^2)$$

$$\therefore \quad y'=\frac{y(5x-4y)}{5x^2+xy+y^2}$$

alternative 2

Take ln of both sides:

$$\frac{x}{y}=\ln(5x+y)$$

Differentiate:

$$\frac{y.1-xy'}{y^2}=\frac{5+y'}{5x+y}$$

Same as above

TA: look out for alternative equivalent forms of answer.

...to be continued...

(25) **Q3.** Obtain f'(x) and simplify as far as possible if :

(12) a) $f(x) = \tan(3\sin x)$

(13) b) $f(x) = \ln(\frac{x+2}{x-2})$

a) $f'(x) = \sec^2(3\sin x) \cdot \frac{d}{dx}(3\sin x)$

$= \sec^2(3\sin x) \cdot 3 \cdot \cos x$

b) Brute force:

$f'(x) \frac{1}{\left(\frac{x+2}{x-2}\right)} \cdot \frac{d}{dx}\left(\frac{x+2}{x-2}\right)$

$= \frac{1}{\cancel{\left(\frac{x+2}{x-2}\right)}} \cdot \frac{(x-2)(1) - (x+2).1}{(x-2)^{\cancel{2}}}$

$= \frac{x-2-x-2}{(x+2)(x-2)}$

$= \boxed{\frac{-4}{x^2-4}}$

or $\frac{-4}{(x+2)(x-2)}$ or look ~~for~~ out for other equivalent + valid forms.

Note $f(x) = \ln(x+2) - \ln(x-2)$

$f'(x) = \boxed{\frac{1}{x+2} - \frac{1}{x-2}}$ actually this is perfectly acceptable as a "simplified" form.

$= \frac{(x-2) - (x+2)}{(x+2)(x-2)}$

= same above

...to be continued...

(25) **Q4.**

(15) **a)** Consider the following exchange between two students A and B:

A: "Since $\lim_{h\to 0} \frac{\cos h}{h} = \infty$ and $\lim_{h\to 0} \frac{1}{h} = \infty$, it follows that $\lim_{h\to 0} \frac{\cos h - 1}{h} = \infty - \infty = 0$."

B: "No, neither $\lim_{h\to 0} \frac{\cos h}{h}$ nor $\lim_{h\to 0} \frac{1}{h}$ is defined. Even if both were positive infinite limits as you say, it is incorrect to say that in general the difference between two infinite limits is zero. Let me illustrate. We know that $\lim_{h\to 0^+} \frac{\cos h}{h} = \infty$ and $\lim_{h\to 0^+} \frac{2}{h} = \infty$, but we know that $\lim_{h\to 0^+} \frac{\cos h - 2}{h} \neq 0$ because we know that $\lim_{h\to 0^+} \frac{\cos h - 1}{h} = 0$."

Answer the following questions:

(3 marks) $\lim_{h\to 0} \frac{\cos h}{h} = \infty$ True or false?

If false, does the limit $\lim_{h\to 0} \frac{\cos h}{h}$ exist? If so, what is it? If not, why not?

False because $\lim_{h\to 0^+} \frac{\cos h}{h} = +\infty$, $\lim_{h\to 0^-} \frac{\cos h}{h} = -\infty$ $\therefore$ no two sided limit

require this for 3 marks

(2 marks) $\lim_{h\to 0^+} \frac{\cos h}{h} = \infty$ True or false? If false, explain why.

True 2

(10 marks) Prove that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$. [Note: L'hôpital's Rule not permitted. No credit.]

[Hint: you may assume $\lim_{h\to 0} \frac{\sin h}{h} = 1$. A trig identity may be involved.]

$$\text{req. limit} = \lim_{h\to 0} \frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1}$$

$$= \lim_{h\to 0} \frac{\cos^2 h - 1}{h \cdot (\cos h + 1)} = \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \frac{(-\sin h)}{(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{-\sin h}{(\cos h + 1)}$$

$$= 1 \cdot \frac{0}{2} = 0$$

...to be continued...

(10) **b)** Show that the following equation has at least one real root: $1 - x^2 = \sin(x)$. [Hint: you will need to use the Intermediate Value Theorem. For full credit, make sure you explain what this theorem says.]

important. closed interval

I.V.T: " If $f(x)$ is continuous on $[a,b]$, where $f(a) \neq f(b)$; and if N lies between $f(a)$ & $f(b)$, there must exist a value $c \in (a,b)$ such that $f(c) = N$. "

important. important

4 marks for correct + full statement.

Let $f(x) = 1 - x^2 - \sin x$.

To say that $1 - x^2 = \sin x$ has a real root is same as saying that $f(x) = 0$ has a real root, i.e. there is a value of x such that $f(x) = 0$.

First, note that $f(x)$ is a continuous function for all real x.

important: 2 marks.

Next, our task is to find 2 values of x such that $f(x)$ changes sign, i.e. one value $f(x) > 0$, another $f(x) < 0$.

TA: Many other choices are possible.

Try $x = 0$, $f(0) = 1 - 0^2 - \sin(0) = 1 > 0$

and $x = \pi$, $f(\pi) = 1 - \pi^2 - \sin(\pi) = 1 - \pi^2 < 0$

$\uparrow$ 0

2 marks for correct choices.

Apply IVT: Since $f(x)$ is continuous on $[0, \pi]$ and $f(0) > 0$, $f(\pi) < 0$

$\therefore$ there must be a value c between 0 & π such that $f(c) = 0$

2 marks for apply IVT correctly.

END OF TEST