SOLUTION & MARKING SCHEME

Surname	Given Name	Student Number

MATH1013M 3.0 Test no. 1 W2014

Time: 45 minutes Value: 100 No calculators or written material allowed.

Attempt as many questions as you please. Maximum is 100. Marks are indicated for each question or part thereof. Part marks will be awarded for relevant and correct intermediate work. Do not hand in additional sheets. If you write overleaf, material will be ignored unless you clearly indicate that this material should be taken into consideration.

(Marks)

(25) Q1.

(10) a) Find the domain and range of $f(x) = \ln(-x^3)$.

(15) b) Show all steps leading to $f^{-1}(x)$. Find the domain and range of $f^{-1}(x)$.

a) $\ln(\ldots)$ must > 0

i.e. $-x^3 > 0$ (5)

or $\boxed{x < 0}$ ← Domain

Range is $\boxed{f(x) \in \mathbb{R}}$ (5), i.e. $\ln(\ldots)$ can be any real number.

b) As per text – 3 step process

Step 1 Let $y = f(x) = \ln(-x^3)$

Step 2 Solve for x $e^y = e^{\ln(-x^3)} = -x^3$

$\therefore x^3 = -e^y$

$\therefore x = -(e^y)^{1/3}$ or $\sqrt[3]{e^y}$

Above is $f^{-1}(y)$, i.e. $f^{-1}(y) = -(e^y)^{1/3}$ or $-e^{y/3}$

$\therefore \boxed{f^{-1}(x) = -e^{x/3}}$

or equivalent $-\sqrt[3]{e^x}$

Must show correct steps for full credit 10 marks

Domain of $f^{-1}(x)$ is identical to range of f, $\boxed{\text{i.e. } \mathbb{R}}$ (2)

Range . . . . domain . . . i.e. $\boxed{f^{-1}(x) < 0}$ (3)

(25) **Q2.** In this question, do **not** use a table of values to suggest a limit. Use appropriate laws or rules to evaluate the following limits if they exist, explaining your reasoning as you proceed. Correct answer without valid reasoning will not receive full merit. If a limit does not exist, explain why not.

(8) a) $\lim_{t \to 16} \dfrac{t - 16}{\sqrt{t} - 4}$

(8) b) $\lim_{x \to -4^-} \dfrac{x}{[[x]]}$ (Reminder: [[x]] is the largest integer value not exceeding x.)

(9) c) $\lim_{u \to \infty} 4u - \sqrt{16u^2 - 64u}$

a) alternative 1: factorization

required limit $= \lim_{t \to 16} \dfrac{(\sqrt{t}-4)(\sqrt{t}+4)}{\sqrt{t}-4}$

$= \sqrt{16} + 4 = \boxed{8}$ 8

alt. 2: rationalization

req. limit $= \lim_{t \to 16} \dfrac{(\cancel{t-16})}{(\sqrt{t}-4)} \dfrac{(\sqrt{t}+4)}{(\sqrt{t}+4)}$ $\cancel{t-16}$

= same as before

b) $\lim_{x \to -4^-} \dfrac{x}{[\![x]\!]} = \dfrac{\lim_{t \to -4^-} x}{\lim_{t \to -4^-} [\![x]\!]} = \dfrac{-4}{-5} = \boxed{\dfrac{4}{5}}$ 8

limit of a quotient = quotient of limits, provided both limits exist, and denominator ≠ 0

c) required limit $= \lim_{u \to \infty} \left(4u - \sqrt{16u^2 - 64u}\right) \cdot \dfrac{4u + \sqrt{16u^2 - 64u}}{4u + \sqrt{16u^2 - 64u}}$

$= \lim_{u \to \infty} \dfrac{16u^2 - (16u^2 - 64u)}{4u + \sqrt{16u^2 - 64u}} = \lim_{u \to \infty} \dfrac{64\cancel{u}}{4\cancel{u} + \sqrt{\frac{16u^2}{u^2} - \frac{64u}{u^2}}}$

$= \dfrac{64}{4 + \sqrt{16 - 0}} = \dfrac{64}{4+4}$

$= \boxed{8}$ 9

TA: in a) + c) allow alternative approaches, eg. L'hospital's Rule

(25) **Q3.** In this question, you are required to give exact values. Simplify as far as possible. For full credit you must give correct reasoning in addition to answers.

(13) a) If $\ln(3x-1) < 0$, solve for x.

(12) b) Evaluate $\log_3 9 - \log_2 16$

a) $\ln(3x-1) < 0$

$\Rightarrow e^{\ln(3x-1)} < e^0$

or $3x - 1 < 1$

or $3x < 2$

or $x < 2/3$

but note: domain of $\ln(3x-1)$

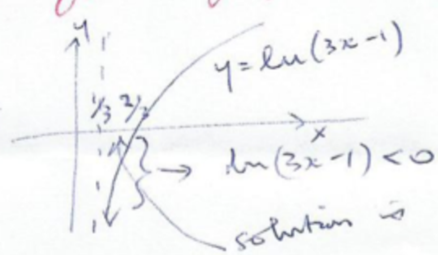
$\Rightarrow 3x - 1 > 0$

or $x > 1/3$

Answer is $\boxed{1/3 < x < 2/3}$

Correct intermediate work is required for full credit.

Alternatively, use graphical approach:



Not required.

b) $\log_3 9 - \log_2 16$

$= \log_3(3^2) - \log_2(2^4)$

$= 2 - 4 = -2$

alternatively

$\log_3 9 = \dfrac{\log_a 9}{\log_a 3}$ ← any base

$= \dfrac{\log_a 3^2}{\log_a 3}$

$= \dfrac{2 \cancel{\log_a 3}}{\cancel{\log_a 3}} = 2$

Same as before

Similarly $\log_2 16 = \cdots = 4$.

(25) Q4. Consider the function $y(x) = \dfrac{x-3}{x-4}$, and answer the following questions.

(9) a) Evaluate the following limits if they exist. If any limit does not exist, explain why: $\lim\limits_{x\to\infty} y$, $\lim\limits_{x\to 4^-} y$, $\lim\limits_{x\to 4^+} y$.

(6) b) With help from part a), what are horizontal and vertical asymptotes, if they exist, to the curve $y(x) = \dfrac{x-3}{x-4}$?

(5) c) State the Intermediate Value Theorem.

(5) d) Consider this assertion: "Since y(3)=0 and y(5)=2, the Intermediate Value Theorem guarantees that there is a value of x between 0 and 2 such that y(x)=1." Do you agree with this assertion? If not, explain why the assertion is incorrect.

a) $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} \dfrac{1-3/x}{1-4/x} = \dfrac{\lim\limits_{x\to\infty} 1-\frac{3}{x}}{\lim\limits_{x\to\infty} 1-\frac{4}{x}} = \dfrac{1-0}{1-0} = \boxed{1}$ 3

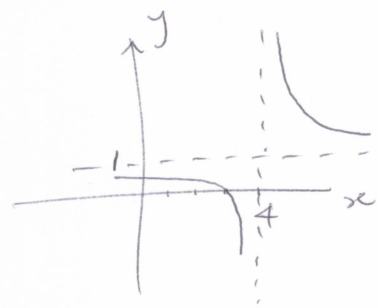
$\lim\limits_{x\to 4^-} y = \lim\limits_{x\to 4^-} \dfrac{4-3}{x-4} = \dfrac{1}{\lim\limits_{x\to 4^-}(x-4)} = \boxed{-\infty}$ 3

$\lim\limits_{x\to 4^+} y = \ldots\ldots = \boxed{+\infty}$ 3

b) as $x\to\infty$, $y\to 1$ $\therefore$ $y=1$ is horizontal asymptote

as $x\to 4^-$, $y\to -\infty$
as $x\to 4^+$, $y\to +\infty$ $\therefore$ $x=4$ is vertical asymptote

Not required:



plot of $y = \dfrac{x-3}{x-4}$

c) IVT: "If f is continuous on interval [a,b] and if N is any number between f(a) and f(b) ($f(a) \neq f(b)$) then there exists at least one number $c \in (a,b)$ such that $f(c) = N$."

important

important

5

TA: must have correct open vs closed brackets for full credit

d) Assertion is incorrect because IVT does not apply here, since y(x) is not continuous on interval [3, 5]

important

5

(See diagram above showing discontinuity in interval [3,5])

END OF TEST