Eigenvalues & Eigenvectors

Definition: Let A be an $n \times n$ (square) matrix.

A scalar λ (lambda) is called an **eigenvalue** of A if there is a nonzero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}.$$

Such a vector $\vec{x}$ is called an **eigenvector** of A corresponding to λ.

Example: The vector $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ since

$$A\vec{x} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\vec{x}$$

The number $\lambda = 4$ is the corresponding eigenvalue.

Example: Suppose, instead, that we know that $\lambda = 4$ is an eigenvalue of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Then

$$A\vec{x} = 4\vec{x}$$

for some vector $\vec{x}$.

Note that this means that

$$(A - 4I)\vec{x} = \vec{0}$$

Now

$$A - 4I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Solutions to $(A - 4I)\vec{x} = \vec{0}$ are given by

$$\vec{v} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4$.

The set of all eigenvectors corresponding to an eigenvalue λ of an $n \times n$ matrix A is just the set of all nonzero vectors in the null space of $A - \lambda I$.

Definition:

Let A be an $n \times n$ matrix and let λ be an eigenvalue. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the **eigenspace** of λ, and is denoted E_λ.

(The eigenspace of λ is the null space of $A - \lambda I$.)

Example: The matrix

$$A = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

has $\lambda = 6$ as an eigenvalue. Find the corresponding eigenspace.

Solution:

$$A - 6I = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let E_6 be the set of solutions to $(A - 6I)\vec{x} = \vec{0}$. Then

$$\begin{aligned} E_6 &= \left\{ \begin{bmatrix} -x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &= \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Hence the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the eigenspace E_6.

Eigenvalues & Eigenvectors of $n \times n$ Matrices

Now that we have defined the determinant of an $n \times n$ matrix, we return to the topic of eigenvalues and eigenvectors.

The eigenvalues of a square matrix A are the solutions of the equation

$$\det(A - \lambda I) = 0.$$

Definitions:

The polynomial $\det(A - \lambda I)$ in λ is called the

characteristic polynomial.

The **equation** $\det(A - \lambda I) = 0$ is called the

characteristic equation.

Procedure for finding the eigenvalues and eigenvectors (eigenspaces) of an $n \times n$ matrix A:

1. Compute the characteristic polynomial, $\det(A - \lambda I)$.

2. Find the eigenvalues by finding solutions λ for the characteristic equation $\det(A - \lambda I) = 0$.

3. For **each** eigenvalue λ, find the null space of $A - \lambda I$.

   This is the eigenspace E_λ, the non zero vectors of which are the eigenvectors of A corresponding to λ. (There are infinitely many eigenvectors for each eigenvalue.)

4. Find a basis for each eigenspace.

Example: Find the eigenvalues and eigenspaces of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(This example is over 3 pages.)

The characteristic polynomial is

$$\begin{aligned}
&\det(A - \lambda I) \\
&= \left| \begin{matrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{matrix} \right] \\
&= (1-\lambda) \begin{vmatrix} -1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 2 \begin{vmatrix} -1 & 1 \\ 0 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\left[(-1-\lambda)(1-\lambda) - 1\right] \\
&\qquad -2[(-1)(1-\lambda) - 0] \\
&= (1-\lambda)[-1+\lambda-\lambda+\lambda^2-1] \\
&\qquad -2[\lambda - 1]
\end{aligned}$$

There are now two options: either expand $\det(A-\lambda I)$ completely and then factor it, or look for common factors earlier to simplify the factorization.

Method 1: expand $\det(A-\lambda I)$ completely

$$\begin{aligned}
\det(A-\lambda I) &= (1-\lambda)[-1+\lambda-\lambda+\lambda^2-1]-2[\lambda-1] \\
&= (1-\lambda)[\lambda^2-2]-2\lambda+2 \\
&= -\lambda^3+\lambda^2+2\lambda-2-2\lambda+2 \\
&= -\lambda^3+\lambda^2 \\
&= \lambda^2(-\lambda+1) \\
&= \lambda^2(1-\lambda)
\end{aligned}$$

Method 2: look for common factors

$$\begin{aligned}
\det(A-\lambda I) &= (1-\lambda)[-1+\lambda-\lambda+\lambda^2-1]-2[\lambda-1] \\
&= (1-\lambda)(\lambda^2-2)+2(1-\lambda) \\
&= (1-\lambda)[(\lambda^2-2)+(2)] \\
&= (1-\lambda)\lambda^2
\end{aligned}$$

The characteristic equation is $(1-\lambda)\lambda^2=0$.

The eigenvalues are $\lambda=1$ and $\lambda=0$.

Note that $\lambda=1$ is a simple root (occurs only once), and $\lambda=0$ is a multiple root.

Label the eignevalues

$$\lambda_1=0,\ \lambda_2=\lambda_3=1$$

Eigenvectors corresponding to $\lambda_1 = 1$:

(null space of $A - \lambda_1 I = A - 1I$)

$$A - 1I = \begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} \in \text{ null space of } A - 1I \quad \therefore E_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Eigenvectors corresponding to $\lambda_2 = \lambda_3 = 0$:

(null space of $A - \lambda_2 I = A - 0I = A$)

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} \in \text{ null space of } A - 0I \quad \therefore E_2 = \text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

The **Fundamental Theorem of Algebra** says that every $n \times n$ matrix A has n eigenvalues.

These eigenvalues are NOT necessarily distinct!

In our example above, we had $\lambda = 1$ with (algebraic) multiplicity one, and $\lambda = 0$ with (algebraic) multiplicity two.

Each of the eigenspaces had a basis consisting of only one vector. This means that

$$\dim E_1 = \dim E_2 = 1.$$

Calculating and factoring $\det(A - \lambda I)$ is sometimes quite difficult.

Whenever possible, look for common factors to simplify the factorization process.

If possible, avoid multiplying out $\det(A - \lambda I)$ completely.

A few theorems about eigenvalues and eigenvectors:

Theorem:

The eigenvalues of a triangular matrix are the entries on the main diagonal.

Theorem:

A square matrix A is invertible	$\iff$	0 is NOT an eigenvalue of A

Theorem:

Let A be an $n \times n$ matrix.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be **distinct** eigenvalues of A with corresponding eigenvectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$.

Then $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ are linearly independent.

(Notice we may have $m < n$.)

Exercise: Find the eigenvalues and eignespaces for

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

For practice with finding common factors, calculate $\det(A - \lambda I)$ by definition, that is, by cofactor expansion along the top row.

The shortest method to calculate $\det(A - \lambda I)$ is by cofactor expansion down the second column.

Answer

The eigenvalue is $\lambda = 3$ with algebraic multiplicity 3

$$E_3 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Similar Matrices & Diagonalization

Definition: Let A and B be $n \times n$ matrices.
We say that A **is similar to** B if there is an invertible matrix P such that

$$P^{-1}AP = B$$

If A is similar to B, we write $A \sim B$.

Note: Try to avoid confusing this with

"A is row equivalent to B"

Note that if A is similar to B, then

$$P^{-1}AP = B \implies AP = PB$$

so we check that $AP = PB$ to check our solution.

This is much shorter than calculating P^{-1} and then checking that $P^{-1}AP = B$!

Example: The matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ is similar to the matrix $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ since

$$AP = PB = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} \text{ with } P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Note: We rarely need to calculate P^{-1} explicitly.

Theorem: Let A, B, and C be $n \times n$ matrices.

Then

1. $A \sim A$
2. If $A \sim B$, then $B \sim A$
3. If $A \sim B$ and $B \sim C$, then $A \sim C$

Theorem: Let A and B be $n \times n$ matrices with $A \sim B$.

Then

1. $\det A = \det B$
2. A is invertible $\iff$ B is invertible
3. A and B have the same rank
4. A and B have the same characteristic polynomial
5. A and B have the same eigenvalues

Definition: An $n \times n$ matrix A is **diagonalizable** if there is a diagonal matrix D such that A is similar to D.

That is, if there is a diagonal matrix D and an invertible matrix P such that

$$P^{-1}AP = D$$

Example: The matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is diagonalizable.

Take $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$

Exercise: Check that $AP = PD$.

Theorem: Let A be $n \times n$ matrix.

Then

$$A \text{ is diagonalizable} \iff \begin{array}{c} A \text{ has } n \text{ linearly} \\ \text{independent eigenvectors} \end{array}$$

That is, there exists an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D$$

if and only if the columns of P are n linearly independent eigenvectors of A, and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

Example: In our example above, we had

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues of A are $\lambda = 4$ and $\lambda = -1$, and we have

$$E_4 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \qquad E_{-1} = \text{Span}\left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$$

Example: Let $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & 1 \end{bmatrix}$

The eigenvalues of A are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -2$, and we have

$$E_0 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad E_{-2} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \right\}$$

Then $P^{-1}AP = D$ with

$$P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

To check this, verify that $AP = PD$.

Exercise: Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, and check your answer.

There are several possible answers, depending on what order you pick for your eigenvalues (which are 1, 2, -1) and which eigenvectors you pick.

One solution is:

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Diagonalization helps us detemine powers of a matrix.

If A is similar to a diagonal matrix D, then there is an invertible matrix P such that

$$P^{-1}AP = D$$

Rearranging this gives

$$AP = PD \text{ and, in fact, } A = PDP^{-1}.$$

Note that

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}.$$

Then

$$A^k = \left(PDP^{-1}\right)^k = PD^kP^{-1}$$

Calculating powers of diagonal matrices is easy:

$$\text{if } D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{ then } D^k = \begin{bmatrix} {a_{11}}^k & 0 & \cdots & 0 \\ 0 & {a_{22}}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {a_{nn}}^k \end{bmatrix}$$

Example: In out previous exercise, $P^{-1}AP = D$ with

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Then

$$A^3 = PD^3P^{-1}$$

Exercise: Let $A = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) For each eigenvalue found in (a), calculate the eigenspace.

(c) Find a diagonal matrix D and an invertible matrix P such that A is similar to D, that is, $P^{-1}AP = D$.

(d) Calculate P^{-1}.

(e) Find A^4, using the information from (c).

Complex Eigenvalues & Eigenvectors

Complex eigenvalues occur in conjugate pairs.

That is, if $\lambda = a + bi$ $(b \neq 0)$ is an eigenvalue of a real matrix A, then

$$\overline{\lambda} = a - bi$$

is **also** an eigenvalue of A.

Moreover, if $\vec{x}$ is an eigenvector of A corresponding to $\lambda = a + bi$ then

$\overline{\vec{x}}$ is an eigenvector of A corresponding to $\overline{\lambda} = a - bi$

Example: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)(1-\lambda) - (-1)(1) \\
&= 1 - 2\lambda + \lambda^2 + 1 \\
&= \lambda^2 - 2\lambda + 2
\end{aligned}$$

Roots of $\lambda^2 - 2\lambda + 2$ are given by the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

The eigenvalues of A are $\lambda_1 = 1 + i$ and $\lambda_2 = \overline{\lambda_1} = 1 - i$.

Eigenvector for $\lambda_1 = 1 + i$:

$$\begin{aligned}
A - (1+i)I &= \begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix} \\
&= \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Vectors in the nullspace look like $\begin{bmatrix} -i\,x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} t$, so $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1 + i$.

Note that $i \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is also an eigenvector for $\lambda_1 = 1 + i$.

Eigenvector for $\lambda_2 = \overline{\lambda_1} = 1 - i$:

The complex conjugate of $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is $\begin{bmatrix} +i \\ 1 \end{bmatrix}$ so $\begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = \overline{\lambda_1} = 1 - i$.

Note that $-i \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is also an eigenvector for $\lambda_2 = \overline{\lambda_1} = 1 - i$.

Exercise 1: Find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$$

Exercise 2: Find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$$

Factoring Polynomials of degree ≥ 3

Quadratic polynomials (degree 2) of the form ax^2+bx+c can be factored using the quadratic formula.

In this course, we have seen the need to factor polynomials of degree more than 3, as we wish to find roots of characteristic equations.

Rational Roots Theorem:

Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

be a polynomial with **integer** coefficients.

If $\frac{a}{b}$ is a root of the polynomial p, that is, if $p\left(\frac{a}{b}\right) = 0$, then

a divides a_0 (a_0 is a multiple of a)
b divides a_n (a_n is a multiple of b)

Example: Let $p(x) = 6x^3 + 13x^2 - 4$

Here $a_0 = -4$ which has factors $\pm 1, \pm 2, \pm 4$
and $a_n = 6$ which has factors $\pm 1, \pm 2, \pm 3, \pm 6$

So if $\frac{a}{b}$ is a root of $x^3 - 2x^2 - 2x - 3$, then the only possibilities for $\frac{a}{b}$ are:

$$\pm\frac{1}{1},\ \pm\frac{1}{2},\ \pm\frac{1}{3},\ \pm\frac{1}{6},\ \pm\frac{2}{1},\ \pm\frac{2}{2},\ \pm\frac{2}{3},\ \pm\frac{2}{6},\ \pm\frac{4}{1},\ \pm\frac{4}{2},\ \pm\frac{4}{3},\ \pm\frac{4}{6}$$

which simplifies to

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm\frac{1}{2},\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{1}{6}$$

(That is, each factor of a_0 divided by each factor of a_n.)

Test each of these possibilities to see if $p\left(\frac{a}{b}\right) = 0$ for that choice of $\frac{a}{b}$.

By trial and error, we find that $x = -2,\ -\frac{2}{3},\ \frac{1}{2}$ are the roots.

Note: a polynomial of degree n has **at most** n roots, so if we find n roots, we may stop.

Not all polynomials of degree n will have n roots, and many may also have repeated roots.

Example: Let $p(x) = x^3 - 2x^2 - 2x - 3$

Here $a_0 = -3$ which has factors $\pm 1, \pm 3$
and $a_n = 1$ which has factors ± 1

So if $\frac{a}{b}$ is a root of $x^3 - 2x^2 - 2x - 3$, then the only possibilities for $\frac{a}{b}$ are:

$$\pm\frac{1}{1},\ \pm\frac{3}{1},\ \text{that is,}\ \pm 1,\ \pm 3$$

$$\begin{aligned} p(1) &= 1 - 2 - 2 - 3 = -6 \\ p(-1) &= -1 - 2 + 2 - 3 = -4 \\ p(3) &= 27 - 18 - 6 - 3 = 0 \ \therefore x = 3 \text{ is a root} \\ p(-3) &= -27 - 18 + 6 - 3 = -42 \end{aligned}$$

In this case, we found only one rational root. Using long division of polynomials, we find that

$$p(x) = x^3 - 2x^2 - 2x - 3 = (x - 3)(x^2 + x + 1)$$

and the polynomial $x^2 + x + 1$ has only complex roots.

Exercises: Factor each of the following characteristic polynomials.

(i) $-\lambda^3 + 2\lambda^2 + 5\lambda - 6$

(ii) $-\lambda^3 + 9\lambda^2 - 27$

(iii) $\lambda^4 - 6\lambda^3 + 9\lambda^2 + 4\lambda - 12$

(iv) $\lambda^4 - 4\lambda^3 + 2\lambda^2 + 4\lambda - 3$

(v) $-\lambda^3 - 3\lambda^2 + 4\lambda - 12$

Note: Since these are characteristic polynomials, they will have n roots. These roots may not all be distinct, so long division may be necessary in order to factor the polynomial.

Solutions:

(i) Roots are -2, 1, 3

(ii) Root is 3

(iii) Roots are -1, 2, 3

(iv) Roots are -1, 1, 3

(v) Roots are