MAT137Y1 - Calculus!
Test 2 – Solutions
November 27th, 2013

1. **Compute the following three limits:**

(a)

$$\lim_{x\to-\infty} \frac{3x^3-2x+1}{4x^3+6x+5} = \lim_{x\to-\infty} \frac{x^3\left[3-\frac{2}{x^2}+\frac{1}{x^3}\right]}{x^3\left[4+\frac{6}{x^2}+\frac{5}{x^3}\right]} = \lim_{x\to-\infty} \frac{3-\frac{2}{x^2}+\frac{1}{x^3}}{4+\frac{6}{x^2}+\frac{5}{x^3}} = \frac{3}{4}$$

(b)

$$\lim_{x\to 0} \frac{3x^3-2x+1}{4x^3+6x+5} = \frac{1}{5}$$

There is no indeterminate form here.

(c)

$$\lim_{x\to\infty} \left(\sqrt{x^2+9x}+\sqrt{x^2+x}\right) = \infty$$

There is no indeterminate form here.

2. **Compute the derivative of $f(x) = 2x^2\arctan(x^2) - \ln(x^4+1)$.**

$$\begin{aligned} f'(x) &= 4x\arctan(x^2) + 2x^2\frac{1}{1+(x^2)^2}2x - \frac{1}{x^4+1}4x^3 \\ &= 4x\arctan(x^2) \end{aligned}$$

3. **Given the function $g(x) = 2\sin(3x)$, calculate $g^{(43)}(0)$.**

I start by computing the first few derivatives:

$$\begin{aligned} g(x) &= 2\sin(3x) \\ g'(x) &= 2 \cdot 3\cos(3x) \\ g''(x) &= -2 \cdot 3^2 \sin(3x) \\ g'''(x) &= -2 \cdot 3^3 \cos(3x) \\ g^{(4)}(x) &= 2 \cdot 3^4 \sin(3x) \end{aligned}$$

Every four derivatives we get back to $\sin(3x)$ with an extra factor of 3^4. Therefore:

$$\begin{aligned} g(x) &= 2\sin(3x) \\ g^{(4)}(x) &= 2 \cdot 3^4 \sin(3x) \\ g^{(8)}(x) &= 2 \cdot 3^8 \sin(3x) \\ &\cdots \\ g^{(40)}(x) &= 2 \cdot 3^{40} \sin(3x) \end{aligned}$$

And then

$$\begin{aligned} g^{(41)}(x) &= 2 \cdot 3^{41} \cos(3x) \\ g^{(42)}(x) &= -2 \cdot 3^{42} \sin(3x) \\ g^{(43)}(x) &= -2 \cdot 3^{43} \cos(3x) \end{aligned}$$

Finally:

$$g^{(43)}(0) = -2 \cdot 3^{43}$$

4. **Calculate the following two limits:**

(a)

$$\lim_{x\to 0} \frac{\cos(2x)-1}{e^{3x}-1-3x} \stackrel{\text{L'H}}{=\!=} \lim_{x\to 0} \frac{-2\sin(2x)}{3e^{3x}-3} \stackrel{\text{L'H}}{=\!=} \lim_{x\to 0} \frac{-4\cos(2x)}{9e^{3x}} = \frac{-4}{9}$$

(b) $\lim_{t\to 1} \left(\frac{2t+3}{3t+2}\right)^{1/(t-1)}$. This is an indeterminate form of type 1^∞.

Let us call $y = \left(\frac{2t+3}{3t+2}\right)^{1/(t-1)}$

We want to compute $\lim_{t\to 1} y$. Let us compute first $\lim_{t\to 1}(\ln y)$.

$$\lim_{t\to 1}(\ln y) = \lim_{t\to 1} \frac{\ln \frac{2t+3}{3t+2}}{t-1} = \lim_{t\to 1} \frac{\ln(2t+3)-\ln(3t+2)}{t-1}$$

$$\stackrel{\text{L'H}}{=\!=} \lim_{t\to 1} \frac{\frac{2}{2t+3} - \frac{3}{3t+2}}{1} = \frac{-1}{5}$$

Hence $\lim_{t\to 1} y = e^{-1/5}$.

5. **Find the slope of the line tangent to the graph of** $x^3 + x^2 y = y^3 + 8$ **at** $(2, 2)$**.**

We differentiate implicitly, thinking of y as a function of x:

$$\frac{d}{dx}\left[x^3 + x^2 y\right] = \frac{d}{dx}\left[y^3 + 8\right]$$

$$3x^2 + 2xy + x^2\frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

When $x = 2$ and $y = 2$, we get $12 + 8 + 4\frac{dy}{dx} = 12\frac{dy}{dx}$.

This solves to $\frac{dy}{dx} = \frac{5}{2}$, which is the answer.

6. **Let f and g be two differentiable functions. We define a new function h by $h(x) = f(x)^{g(x)}$. Find a formula for $h'(x)$ in terms of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$.**

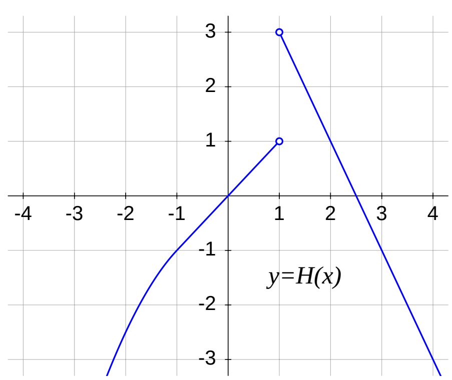
$$\begin{aligned} \ln h(x) &= g(x) \ln f(x) \\ \frac{d}{dx}[\ln h(x)] &= \frac{d}{dx}[g(x) \ln f(x)] \\ \frac{h'(x)}{h(x)} &= g'(x) \ln f(x) + g(x)\frac{f'(x)}{f(x)} \end{aligned}$$

Finally, we solve for $h'(x)$:

$$h'(x) = f(x)^{g(x)} \left[g'(x) \ln f(x) + g(x)\frac{f'(x)}{f(x)} \right]$$

7. **Calculate the limits indicated from the graphs of the functions.**

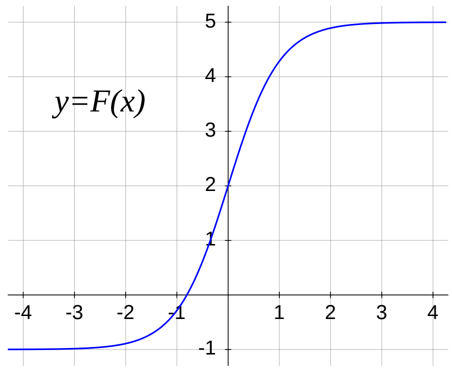
(a) $\lim_{x \to 0} \dfrac{H(x)}{H(2+3x)-1}$



From the graph, this is an indeterminate form of type $\frac{0}{0}$, so we can use L'Hôpital:

$$\lim_{x \to 0} \frac{H(x)}{H(2+3x)-1} \overset{\text{L'H}}{=\!=} \lim_{x \to 0} \frac{H'(x)}{3H'(2+3x)} = \frac{H'(0)}{3H'(2)} = \frac{1}{3(-2)} = \frac{-1}{6}$$

(b) $\lim_{x \to 2} \dfrac{F^{-1}(x)}{x-2}$

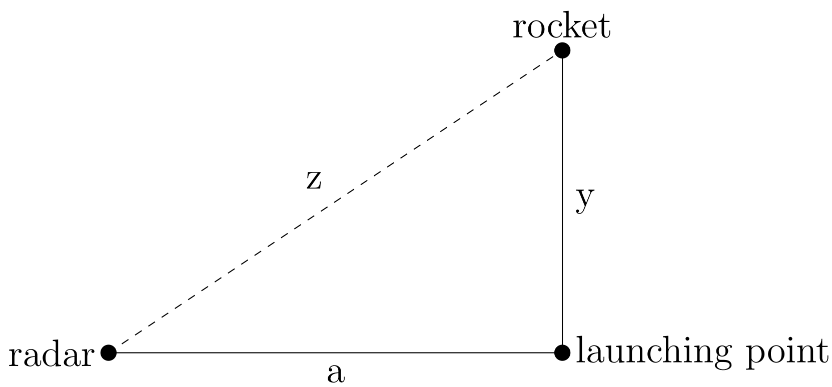


From the graph, $F^{-1}(2) = 0$. Hence

$$\lim_{x \to 2} \frac{F^{-1}(x)}{x-2} = \lim_{x \to 2} \frac{F^{-1}(x) - F^{-1}(2)}{x-2} = \left(F^{-1}\right)'(2) = \frac{1}{F'(0)} = \frac{1}{3}$$

This can also be solved using L'Hôpital's Rule instead.

8. **A rocket that is launched vertically is tracked by a radar station located on the ground 3 kilometres from the launch site. What is the vertical speed of the rocket at the instance that its distance from the radar is 5 kilometres and this distance is increasing at the rate of 5000 km/h?**



Let us call a the distance from the radar station to the launching point. $a = 3km$ is constant.

Let y be the height of the rocket and z be the distance from the radar station to the rocket. Both y and z are functions of time. We want to compute $\frac{dy}{dt}$ (the speed of the rocket) at the time when $\frac{dz}{dt} = 5000km/h$ and $z = 5km$. Notice that, at this instance $y = \sqrt{z^2 - a^2} = 4km$.

We differentiate implicitly the equation $a^2 + y^2 = z^2$ with respect to time.

$$0 + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

Hence

$$\frac{dy}{dt} = \frac{z}{y}\frac{dz}{dt} = \frac{5}{4}5000km/h = 6250km/h$$

9. **Let f be a differentiable function with domain $(-\infty, \infty)$.**

(a) **If f is periodic, does f' have to be periodic?** Yes.

We assume that there is a number $T > 0$ such that $f(x+T) = f(x)$ for all x. That is what it means for f to be periodic. I want to show that f' satisfies the same equation. I take derivatives on both sides and use the chain rule:

$$\begin{aligned} \frac{d}{dx} f(x+T) &= \frac{d}{dx} f(x) \\ f'(x+T)\frac{d}{dx}(x+T) &= f'(x) \\ f'(x+T) \cdot 1 &= f'(x) \\ f'(x+T) &= f'(x) \end{aligned}$$

Hence, f' is periodic as well.

(b) **If f' is periodic, does f have to be periodic?** No.

For example, $f(x) = x + \sin x$ is not periodic, but $f'(x) = 1 + \cos x$ is periodic.

10.

(a) **Give the formal definition of the statement $\lim_{x\to\infty} f(x) = L$.**

Let f be a function defined at least on an open interval (p, ∞) for some real number p. We say that

$$\lim_{x\to\infty} f(x) = L$$

when for every $\varepsilon > 0$, there exists $M \in \mathbb{R}$ such that

$$\text{if} \quad x > M, \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

(b) **Prove, using the formal definition, that** $\lim_{x\to\infty} \frac{x+2}{x+3} = 1$**.**

- *Preparation.*
  I need to find a relation between $|f(x) - 1|$ and x.

  $$|f(x) - 1| \;=\; \left|\frac{x+2}{x+3} - 1\right| \;=\; \left|\frac{-1}{x+3}\right| \;=\; \frac{1}{x+3}$$

  If I want $|f(x) - 1| < \varepsilon$, I can obtain it by requiring $x + 3 > \frac{1}{\varepsilon}$.
  In particular, if $x > \frac{1}{\varepsilon}$ this will be true. This suggests $M = \frac{1}{\varepsilon}$.
- *The actual proof.*
  Let $\varepsilon > 0$. I take $M = \frac{1}{\varepsilon}$.
  Now, if $x > M$, we have that

  $$|f(x) - 1| = \frac{1}{x+3} < \frac{1}{x} < \frac{1}{M} = \varepsilon$$

  as desired.