NAME marking Scheme, **ID**

no aids allowed, time allowed: 100 min. total mark: 50 (+5 bonus marks)

1. (10 marks) Here is all the information about a LPP in the standard format as well as the matrix B^{-1} that is responsible for bringing the simplex method to certain stage with x_1 and x_3 as basic variables:

$$A = \begin{bmatrix} 1 & 4 & 2 & 4 \\ 0 & 1 & 2 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix}$$

Use this information to determine the feasible solution at this stage (together with the value of the objective function) and determine whether this feasible solution is optimal. (You must use the matrix B^{-1} and must present all the necessary computations.)

$X_B = B^{-1} b = \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$ (1.5)

$Z = C_B^T X_B = [1 \ 1] \begin{bmatrix} 2 \\ 1/2 \end{bmatrix} = \frac{5}{2}$ (1.5) (1)

to decide if The Sln is optimal we need to investigate all The $Z_j - C_j$:

(1) $Z_1 - C_1 = 0 = Z_3 - C_3$ b/c x_1 and x_3 are basic

$Z_2 - C_2 = C_B^T B^{-1} A_2 - C_2 = [1 \ 1] \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} - 2 = [1 \ -1/2] \begin{bmatrix} 4 \\ 1 \end{bmatrix} - 2 =$ (2)

(1) $4 - 1/2 - 2 > 0$

$Z_4 - C_4 = [1 \ -1/2] \begin{bmatrix} 4 \\ 2 \end{bmatrix} - 3 = 0 \geq 0$ so, $\boxed{\text{yes The Sln is optimal.}}$

(1)

please give no mark if they Solve The problem independently of B^{-1} and gave some answers

2. (10 marks) Here is the coefficient and constraint matrices of a LPP in the canonical format:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Use phase I of the two phase method to present an initial tableau for the simplex method (this tableau should be ready to start phase II.)

add one artificial variable y : and maximize $Y = -y$ (1.5)

	x_1	x_2	x_3	y	Y	
x_1	1	2	4	0	0	3
y	0	2	2	1	0	2
	0	0	0	+1	1	0

re combine to get rid of (1.5)

$-R_2 + R_3$

		x_2			Y	
x_1	1	2	4	0	0	3
y	0	2	2	1	0	2
	0	−2	−2	0	1	−2

↑ enters x_2

departs y (1.5)

	x_1	x_2	x_3	y	Y	b
x_1	1	0	2	−1	0	1
x_2	0	1	1	½	0	1
	0	0	0	1	1	0

end of phase I

→ give them 5 marks if they did all of above correctly

but to get to the initial tableau for phase II: new objective row

	x_1	x_2	x_3	z	b
x_1	1	0	2	0	1
x_2	0	1	1	0	1
	−1	−2	−3	1	0

(2)

(2) recombine to clear &

$R_1 + R_3$
$2R_2 + R_3$

x_1	1	0	2	0	1
x_2	0	1	1	0	1
	0	0	0	1	3

(1.5)

tableau ready for phase II & indeed optimal.

Note if they choose two artificial variables it would take longer but still same marking scheme.

3. (12 marks) Assume x_0 and w_0 are feasible solutions to the primal (in the standard format), and the dual respectively. Assume also that $c^T x_0 = b^T w_0$. Prove that these two solutions must be optimal solutions to the primal and the dual problems. (please do not rely on quoting other theorems, but present all the necessary details.)

This is Thm 3.6 & 3.4 together.

We need to prove for all feasible slns of the primal x and the dual w, we have $c^T x \le b^T w$.

Since the primal problem is about maximizing 0.5 and

all $c^T x \le b^T w_0 = c^T x_0$ 0.5 Then $c^T x_0$ is max value of the obj funct 1

Similarly the dual problem is about minimizing 0.5, and

$\forall w \quad b^T w \ge c^T x_0 = b^T w_0$ 0.5, so $b^T w_0$ is min value 1 of the obj function.

Now to prove $\forall x, w$ feasible slns $c^T x \le b^T w$: Since x is feasible 1 $Ax \le b$

$Ax_0 \le b \Rightarrow w^T A x \le w^T b = b^T w$ (*)

1 b/c $w_0 \ge 0$ 1 b/c $w_0^T b$ is Scalar

Since w_0 is feasible 1 we have $A^T w \ge c$ or $w_0^T A \ge c^T$ 1

as $x \ge 0$ 1 $w^T A x \ge c^T x_0$ (**)

now (*) & (**) given $c^T x \le w^T A x \le b^T w$. 2

4. (11 marks) write the dual of the following LPP, and use your knowledge of duality to solve the dual problem.

maximize $3x_1 + 4x_2$ subject to

$$x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 7$$
$$2x_1 + x_2 \leq 7,$$
$$x_1 \geq 0,\ x_2 \text{ unrestricted.}$$

The dual problem is min $4w_1 + 7w_2 + 7w_3$ s.t

$w_1 + w_2 + 2w_3 \geq 3$

$w_1 + 2w_2 + w_3 = 4$

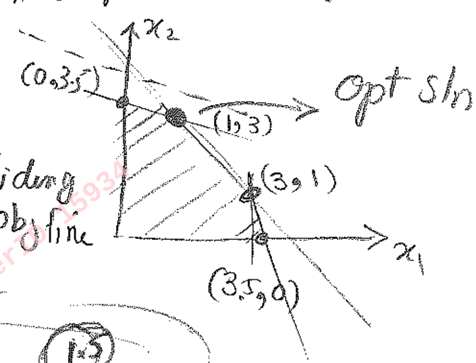
$w_1, w_2 \geq 0$

4

using our knowledge of duality (That is question #3, previous page)

We need to solve the primal (any method)

Simplex — Checking all The extreme pt — Sliding obj line



Sln to The primal is $3 + 12 = 15$ at $(1,3)$ 1.5

So by question #3 if w_0 satisfies $b^T w_0 = 15$ Then it is an optimal sln: $b^T w_0 = [4\ 7\ 7]\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 4w_1 + 7w_2 + 7w_3 = 15$ 2.5

Solve $\begin{cases} 4w_1 + 7w_2 + 7w_3 = 15 \\ w_1 + 2w_2 + w_3 = 4 \\ w_1 + w_2 + 2w_3 \geq 3 \end{cases} \longrightarrow$ $4E_2 - E_1 \Rightarrow w_2 - 3w_3 = 1$

$-E_2 + \text{ineq } 3 \Rightarrow -w_2 + w_3 \geq -1$

3

$w_2 = 1 + 3w_3 \longrightarrow \cancel{-1} - 3w_3 + w_3 \geq \cancel{-1}$ $\quad -2w_3 \geq 0$ so let $w_3 = 0$

as $w_3 \geq 0$

to get $w_2 = 1$ & $w_1 = 2$ so one such sln is $(2, 1, 0)$

5. Short answer, or short calculation questions (you may quote results that you may need for your arguments):

a) (3 marks) To maximize $2x_1 + 3x_2 + x_3$ we operate on the constraint matrix until we reach the optimal solution which looks like $(4, 0, 3, 0, 2, 0)$. Determine the values of (or the range of) the z_j, $j = 1, \dots 6$?

Since the basic variables are x_1, x_3, x_5 (or possibly more)

we know $z_j - c_j = 0$ for $j = 1, 3, 5$ so $z_1 = c_1 = 2$ (1.5) $z_3 = c_3 = 1$

$z_5 = c_5 = 0$

for others $z_j - c_j \geq 0$ so $z_2 \geq c_2 = 3$ & $z_4 \geq c_4 = 0$ $z_6 \geq c_6 = 0$ (1.5)

b) (4 marks) At some stage of the simplex tableau we reach at an extreme point with coordinates $(0, 3, 0, 0, 2, 0)$, which is not optimal. The basic variables are x_2, x_3 and x_5 in this order, and the value of objective row below x_1 is the largest negative with $t_1^T = [-4, -1, 0]$. Discuss how you will proceed (and why?)

	x_1	b	θ
x_2	-4	3	$-3/4$
x_3	-1	0	0
x_5	0	2	undefined

x_1 enters

(1.5) neither of the constraints imposes any restriction on the value of the entering variable. (2)

$-4x_1 + x_2 = 3$ (0.5)
$-x_1 + x_3 = 0$
$0x_1 + x_5 = 2$

x_1 can increase indefinitely without x_i becoming negative $i = 2, 3, 5$

Our obj is now unbounded as x_1 can increase indef and there is $\boxed{\text{no opt sln}}$ (2)

c) (5 marks) Why the number of the non zero-components of an extreme point is at most equal to the number of the constraints (other than $x_j \geq 0$)?

(3) # of non zero components of an extreme pt is corresponds to the # of lin indep columns of matrix A (in the canonical format of the LPP)

This is Th 1.9

# of lin indep column of A = rank A $\leq$ # rows of A.

(2)