## 3.3.2 Parabolic equations

To transform a parabolic equation into its canonical form, we require a change of coordinate acting such that

$$B = C = 0 \quad \text{(in the notation of 3.2)}$$

However since by definition $AC - B^2 = 0$, it is sufficient to require that $C = 0$.

$\Rightarrow$ we need $\quad a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0$

But now recall that $ac - b^2 = 0$ so this is a perfect square so that it can be rewritten as

$$a\left(\eta_x + \sqrt{\frac{c}{a}}\,\eta_y\right)^2 = 0$$

alternatively $\quad \frac{1}{a}\left(a\eta_x + b\eta_y\right)^2 = 0.$

$\Rightarrow$ we can take $\eta$ solution of the first order PDE

$$a\eta_x + b\eta_y = 0$$

$\Rightarrow$ $\eta$ constant on the characteristics defined by $\frac{dy}{dx} = \frac{b}{a}$

[ Note that this time $\xi$ can be any function of $x$ and $y$ such that the Jacobian of $(\xi, \eta)$ doesn't vanish

Example: $\quad x^2u_{xx} - 2xy\,u_{yx} + y^2u_{yy} + xu_x + yu_y = 0$

$$\delta(\mathcal{L}) = x^2y^2 - x^2y^2 = 0$$

The characteristics satisfy $\quad \frac{dy}{dx} = \frac{xy}{x^2} = -\frac{y}{x}$

So $\quad \ln y = -\ln x + \text{const}$

or $\quad y = \frac{K}{x} \quad \Rightarrow$ take $\eta = xy$ and for simplicity, $\xi = x$

$$u_x = u_\xi + y u_\eta = v_\xi + \frac{\eta}{\xi} v_\eta \quad \text{since} \quad \xi_x = 1 \quad \xi_y = 0$$

$$u_y = x u_\eta = \xi v_\eta \qquad \eta_x = y \quad \eta_y = x$$

$$u_{xx} = u_{\xi\xi} + y u_{\xi\eta} + y^2 u_{\eta\eta} = v_{\xi\xi} + \frac{\eta}{\xi} v_{\eta\xi} + \left(\frac{\eta}{\xi}\right)^2 v_{\eta\eta}$$

$$u_{xy} = xy\, u_{\eta\eta} + x\, u_{\eta\xi} + u_\eta = \eta v_{\eta\eta} + \xi v_{\eta\xi} + v_\eta$$

$$u_{yy} = x^2 u_{\eta\eta} = \xi^2 v_{\eta\eta}$$

So we now have

$$\xi^2 \left[ u_{\xi\xi} + 2\frac{\eta}{\xi} u_{\eta\xi} + \frac{\eta^2}{\xi^2} u_{\eta\eta} \right]$$

$$- 2\eta \left[ \eta\, u_{\eta\eta} + \xi u_{\eta\xi} + u_\eta \right]$$

$$+ \frac{\eta^2}{\xi^2} \left[ \xi^2 u_{\eta\eta} \right] + \xi \left( u_\xi + \frac{\eta}{\xi} u_\eta \right) + \frac{\eta}{\xi} \left( \xi u_\eta \right) = 0$$

$$\Leftrightarrow \quad \xi^2 u_{\xi\xi} + \xi u_\xi = 0 \quad \Leftrightarrow \quad \boxed{u_{\xi\xi} + \frac{1}{\xi} u_\xi = 0}$$

→ the canonical form required.

This is now a simple ODE for $v = \frac{\partial u}{\partial \xi}$

## 3.3.3 Canonical form for Elliptic equations

Given a second order linear PDE which is elliptic, to reduce it to its canonical form we must find a coordinate change $(x,y) \to (\xi,\eta)$ such that

$$\begin{cases} A = C \\ B = 0 \end{cases}$$

So we need
$$\begin{cases} a\xi_x^2 + 2b\,\xi_x\xi_y + c\,\xi_y^2 = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \\ a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0 \end{cases}$$

let's construct the complex quantity $\phi = \xi + i\eta$ then this system is equivalent to

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0$$

Indeed

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = a(\xi_x + i\eta_x)^2 + 2b(\xi_x + i\eta_x)(\xi_y + i\eta_y) + c(\xi_y + i\eta_y)^2$$

$$= a\xi_x^2 - a\eta_x^2 + 2b(\xi_x\xi_y - 2b\eta_x\eta_y + c\xi_y^2 - c\eta_y^2 + i\left[2a\xi_x\eta_x + 2b(\xi_x\eta_y + \eta_x\xi_y) + 2c\,\xi_y\eta_y\right]$$

So equating real & imaginary parts to 0 recovers the required system.

$\Rightarrow$ Characteristic equations imply

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{ac-b^2}}{a} \qquad \text{since } ac - b^2 < 0$$

however, this time the characteristics "live" in a "complex plane".

The characteristic equations are complex conjugates so their solutions (say $\phi$ and $\psi$) will also be C.Cs.

Once the solution is found, we recover $\xi$ and $\eta$ by taking

$$\xi = Re(\phi)$$
$$\eta = Im(\phi).$$

(Note: we can arbitrarily choose $\phi$ or $\psi$ → the only difference is in the sign of $\eta$).

Example: the Tricomi equation $u_{xx} + xu_{yy} = 0$ for $x > 0$

then we solve

$$\frac{dy}{dx} = \pm i\sqrt{x} \Rightarrow dy = \pm i\sqrt{x}\,dx$$

so the solution is

$$\frac{3}{2}y = \pm i x^{3/2} + \text{constant}$$ → choose constant $= \phi$

so let $\phi = \frac{3}{2}y \pm i x^{3/2}$

so $\begin{cases} \xi = \frac{3}{2}y \\ \eta = x^{3/2} \end{cases}$

then $\begin{cases} \xi_x = 0 & \xi_y = \frac{3}{2} \\ \eta_x = \frac{3}{2}x^{1/2} & \eta_y = 0 \qquad \eta_{xx} = \frac{3}{4}x^{-1/2} \end{cases}$

so
$$u_{xx} + xu_{yy} = \frac{9}{4}x\, u_{\eta\eta} + \frac{3}{4}x^{-1/2}u_\eta + x\left(\frac{9}{4}u_{\xi\xi}\right)$$
$$= 0.$$

$$x = \left(\frac{2}{3}\eta_x\right)^2$$

$$\Rightarrow u_{\eta\eta} + u_{\xi\xi} + \frac{1}{3}x^{-3/2}u_\eta = 0$$

$$\Rightarrow u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta}u_\eta = 0$$ → Canonical form of the equation for $x > 0$

# Review: Elements of Fourier Series

## ① Periodic function

- A periodic function is a function which satisfies the relation

$$f(x) = f(x+T) \quad \text{for all } x \text{, and a given } T > 0$$

$T$ is the period of the function.

- Note that a function which is periodic with period $T$ is also periodic with period $nT$ for any $n \in \mathbb{N}$, $n > 0$. Usually $T$ is the smallest real value for which $f(x) = f(x+T)$ holds.

## ② Orthogonality

- An inner product can be defined for any two functions on an interval $[a,b]$ as

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x)\, dx$$

where $w(x)$ is a fixed positive weight function (usually satisfying $\int_a^b w(x)dx = 1$.)

- Two functions are therefore orthogonal on $[a,b]$ provided $\langle f, g \rangle = 0$.

Property: with $w(x) = \frac{1}{2L}$

- the functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L, L]$ for all $(m, n)$
- the functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L, L]$ for all $m \neq n$
- the functions $\cos\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L, L]$ for all $m \neq n$

## ③ Fourier Series

Any function $f$ periodic with period $2L$ can be written as the series (called a Fourier Series)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

the Fourier coefficients

$$\begin{cases} a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \end{cases}$$

<u>Proof</u>: Let $m>0$

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$= \int_{-L}^{L}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]\cos\left(\frac{m\pi x}{L}\right)dx$$

$$= \int_{-L}^{L} a_0 \cos\left(\frac{m\pi x}{L}\right)dx + \int_{-L}^{L}\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$+ \int_{-L}^{L}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

given some prayer for convergence of the series

$$= \int_{-L}^{L} a_m \cos^2\left(\frac{m\pi x}{L}\right)dx = \frac{2L}{2}a_m = La_m$$

so $$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$ as required

(and similarly for the other terms)

## ④ Properties of Fourier $a_n, b_n$ coefficients

- if $f(x)$ is an even function $(f(x) = f(-x))$ then $b_n = 0 \quad \forall n$

- if $f(x)$ is an odd function $(f(x) = -f(-x))$ then $a_n = 0 \quad \forall n$

- The Fourier coeffs. of the sum of two functions $f$ and $g$ is equal to the sum of the Fourier coefficients

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$$

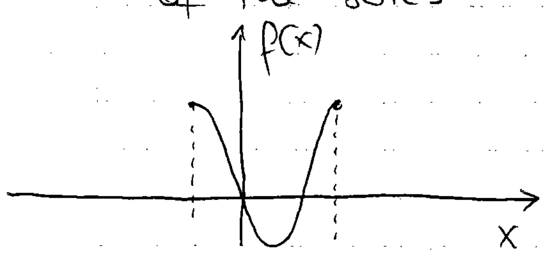
then

$$f+g\ (x) = (a_0 + A_0) + \sum_{n=1}^{\infty} (a_n + A_n) \cos\left(\frac{n\pi x}{L}\right) + (b_n + B_n) \sin\left(\frac{n\pi x}{L}\right)$$

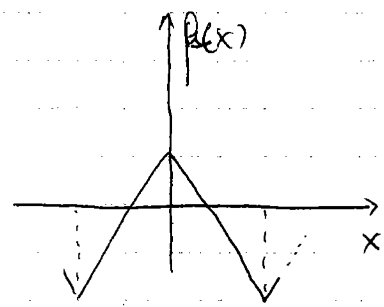
<u>BUT</u> : not true for the product!

- The Fourier series can be differentiated/integrated term by term to obtain the Fourier series of the derivative/integral of a function.

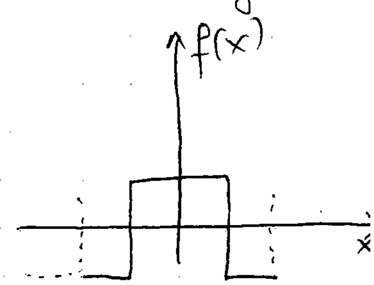
- The smoother the function, the quicker the convergence of the series:




$a_n, b_n \sim \frac{1}{n^3}$ or faster




$a_n, b_n \sim \frac{1}{n^2}$

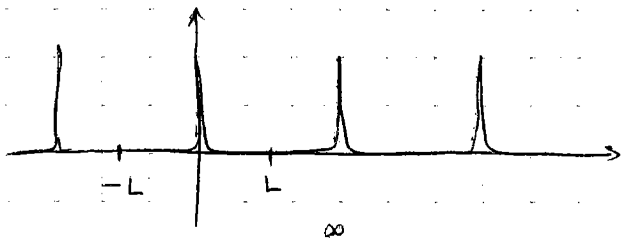



$a_n, b_n \sim \frac{1}{n}$.

Note:

- The function constructed from a Fourier Series may have different discontinuities than the one which it is trying to approximate (see HW).

- The Fourier Series for a set of $\delta$ functions is



$$f(x) = \sum_{n=0}^{\infty} \delta(x-2nL) + \delta(x+2nL)$$

so $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} \delta(x)\,dx = \frac{1}{2L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \delta(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{\cos(0)}{L} = \frac{1}{L}$$

$$b_n = 0$$

so $$f(x) = \frac{1}{2L} + \sum_{1}^{\infty} \frac{1}{L}\cos\left(\frac{n\pi x}{L}\right)$$

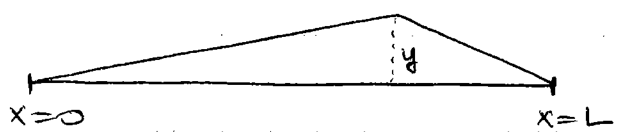
$$f(x) = \frac{1}{2L}\sum_{-\infty}^{+\infty} \cos\left(\frac{n\pi x}{L}\right)$$

CHAPTER 4 Generic behaviour of Hyperbolic/Parabolic/Elliptic equations through examples. Method of separation of variables

## 4.1 The wave equation: the vibrating string

Let a string be tightly stretched with ends attached at $x=0$ and $x=L$.

Displacements away from rest are measured by $y$.




The equation of motion of the string follows a wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with boundary condition $y(0,t)=0$, $y(L,t)=0$.

### Method of separation of variables

Let's assume that we can find a solution with the form

$$y(x,t) = X(x)\,T(t)$$

Then $$X(x)\frac{d^2T}{dt^2} = c^2\,T(t)\frac{d^2X}{dx^2}$$

$$\Leftrightarrow \quad \underbrace{\frac{1}{T(t)}\frac{d^2T}{dt^2}}_{\text{a function of time only}} = \underbrace{\frac{c^2}{X(x)}\frac{d^2X}{dx^2}}_{\text{a function of x only}}.$$