Review: First order (first degree) ODES

I Standard forms

let y be a function of the independent variable x only: $y(x)$

y = dependent variable

First order ODES for $y(x)$ can be written in the equivalent form

$$\frac{dy}{dx} = F(x,y)$$

or $$A(x,y)\,dx + B(x,y)\,dy = 0$$

$$\left(\text{with } F(x,y) = -\frac{A}{B}\right)$$

II Types of equation & method of solution

II.1 Separable variables

Suppose $F(x,y) = f(x)\,g(y)$

or $A(x,y) = a(x)\,\alpha(y)$
$B(x,y) = b(x)\,\beta(y)$

Then

$$\frac{dy}{dx} = F(x,y) \Rightarrow \frac{dy}{g(y)} = dx \cdot f(x)$$

which can be integrated respectively in y and x:

$$\int \frac{dy}{g(y)} = \int dx\, f(x).$$

<u>Example</u> $\quad \dfrac{dy}{dx} = \dfrac{4y}{x(y-3)}$

$$\Rightarrow \quad \frac{y-3}{4y}\,dy = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{y-3}{4y}\,dy = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{4}y - \frac{3}{4}\ln y + K = \ln x$$

arbitrary constant of integration to be fitted to boundary condition

<u>Note</u>: Although a solution was found, it is not always possible to write it out easily as $y(x)$.

I.2 Exact equations

Suppose we try to solve the form

$$A(x,y)dx + B(x,y)dy = 0$$

In some cases, this may be the exact differential of a function $U(x,y)$:

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

if it so happens that $\begin{cases} A(x,y) = \dfrac{\partial U}{\partial x} \\ B(x,y) = \dfrac{\partial U}{\partial y} \end{cases}$ then $dU = 0$

and the solution is simply $U = \text{constant}$.

How do we know this is an exact differential?

If $A = \dfrac{\partial U}{\partial x}$ and $B = \dfrac{\partial U}{\partial y}$ then $\boxed{\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}}$

=> Method :

Given the equation $A(x,y)dx + B(x,y)dy = 0$

① Test if exact : calculate $\frac{\partial A}{\partial y}$ and $\frac{\partial B}{\partial x}$

② If exact then find the function $U(x,y)$ such that

$$A(x,y) = \frac{\partial U}{\partial x}$$
$$B(x,y) = \frac{\partial U}{\partial y}$$

③ Set this function to an arbitrary constant to obtain solution

Example : $y\,dx + x\,dy = 0$ $\quad A(x,y) = y$, $B(x,y) = x$

① $\frac{\partial A}{\partial y} = 1 = \frac{\partial B}{\partial x}$ => this is an exact differential

② What is $U(x,y)$ satisfying

$$\begin{cases} \frac{\partial U}{\partial x} = y & \text{try } U = xy + \text{a function of } y \\ \frac{\partial U}{\partial y} = x & \text{try } U = xy + \text{a function of } x \end{cases}$$

=> $U = xy + \text{constant}$

③ Set $U = \text{const}$ to get solution so

$U(x,y) = K$

=> $xy = K'$ (another constant)

=> $y = \frac{K'}{x}$

II-3 Linear equations

- Linear, first order ODEs can always be written as

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

- Suppose we could find a function $\mu(x)$ such that

$$\mu(x)\frac{dy}{dx} + \mu(x)\,P(x)\,y = \frac{d}{dx}\big(\mu(x)\,y\big) \qquad (*)$$

Then the linear ODE would become

$$\frac{d}{dx}\big(\mu(x)\,y\big) = \mu(x)\,Q(x)$$

which can be formally integrated in x to give the solution

$$\mu(x)\,y(x) - \mu(0)\,y(0) = \int_0^x \mu(x')\,Q(x')\,dx'$$

- To find $\mu(x)$, we need to find a function satisfying $(*)$:

$$\cancel{\mu(x)\frac{dy}{dx}} + \mu(x)\,P(x)\,y = \cancel{\mu(x)\frac{dy}{dx}} + \frac{d\mu}{dx}\,y$$

$$\Rightarrow \quad \mu(x)\,P(x) = \frac{d\mu}{dx}$$

$$\Rightarrow \quad \frac{d\mu}{\mu} = P(x)\,dx$$

$$\Rightarrow \quad \ln\mu = \int P(x)\,dx$$

$$\Rightarrow \quad \boxed{\mu(x) = e^{\int P(x)dx}}$$

So: <u>Method</u>: ① Calculate $\mu(x) = e^{\int P(x)dx}$

② write $\frac{d}{dx}\big(\mu(x)\,y\big) = Q(x)\,\mu(x)$

and integrate it.

Example $\quad \dfrac{dy}{dx} + \dfrac{2-3x^2}{x^3}\, y = 1 \quad \Rightarrow \quad P(x) = \dfrac{2}{x^3} - \dfrac{3}{x}$

$Q(x) = 1$

① $$\mu(x) = e^{\int\left(\frac{2}{x^3} - \frac{3}{x}\right) dx}$$
$$= e^{-\frac{1}{x^2} - 3\ln x}$$
$$= e^{-\frac{1}{x^2}} \cdot \frac{1}{x^3}$$

② so $$\frac{d}{dx}\left(\frac{1}{x^3} e^{-\frac{1}{x^2}} y\right) = \frac{1}{x^3} e^{-\frac{1}{x^2}} \cdot 1$$

integrate in x: (say from $x' = 1$ to x, to avoid singularity at $x = 0$)

$$\frac{1}{x^3} e^{-\frac{1}{x^2}} y(x) - e^{-1} y(1) = \int_1^x \frac{1}{x'^3} e^{-\frac{1}{x'^2}}\, dx'$$
$$= \frac{1}{2}\left[e^{-\frac{1}{x'^2}}\right]_1^x$$
$$= \frac{1}{2}\left(e^{-\frac{1}{x^2}} - e^{-1}\right)$$

so
$$y(x) = e^{-1}\left(y(1) - \frac{1}{2}\right) x^3 e^{\frac{1}{x^2}} + \frac{x^3}{2}$$

II.4 Inexact equations

(A similar idea: see notes in RHB).

II.5 Homogeneous equations

Homogeneous equations are ODES that can be written as
$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$
They are suitably transformed into a simpler form by a change of variables:
$$v = \frac{y}{x} \quad (\text{or } y = vx) \quad \text{where } v = v(x)$$

Indeed: $\dfrac{dy}{dx} = \dfrac{d}{dx}(vx) = x\dfrac{dv}{dx} + v$
$$= F(v)$$

So $\quad x\dfrac{dv}{dx} + v = F(v)$

$\Rightarrow \quad \dfrac{dv}{F(v)-v} = \dfrac{dx}{x}$ which can be integrated in v and in x.

Example $(y-x)\dfrac{dy}{dx} + (2x+3y) = 0$

① Is it homogeneous? Yes: divide by x
$$\left(\frac{y}{x}-1\right)\frac{dy}{dx} + \left(2+\frac{3y}{x}\right) = 0$$
$$\frac{dy}{dx} + \frac{2+3\frac{y}{x}}{\frac{y}{x}-1} = 0$$

② Let $v = \dfrac{y}{x}$ then
$$x\frac{dv}{dx} + v + \frac{2+3v}{v-1} = 0$$
$$x\frac{dv}{dx} + \frac{(v-1)v+2+3v}{v-1} = 0$$
$$x\frac{dv}{dx} + \frac{v^2+2v+2}{v-1} = 0 \rightarrow \underline{\text{Homework}}$$

CHAPTER 2 First order PDEs (in 2 dimensions)

2.1 General formulae

A first order PDE in 2 dimensions is in the form of

$$\mathcal{F}\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

- A first order linear PDE in 2 dimensions is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t)u + d(x,t).$$

- A first order semilinear PDE in 2D is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$$

- A first order quasilinear PDE is

$$a(x,t,u)\frac{\partial u}{\partial t} + b(x,t,u)\frac{\partial u}{\partial x} = c(x,t,u)$$

A fully nonlinear PDE is none of the above!

2.2 Method of characteristics for quasilinear equations

2.2.1 Warmup example

Let's study $u_t = c_0 u + c_1(x,t)$ $\quad$ c_0 constant

Note that for each x, it is actually an ODE in t
→ fix x and solve it!

Use integrating factor method (for example)

$$u_t - c_0 u = c_1(x,t)$$

⇒ We try to find an integrating factor $\mu(x,t)$ such that

$$\mu u_t - \mu C_0 u = \mu G(x,t)$$

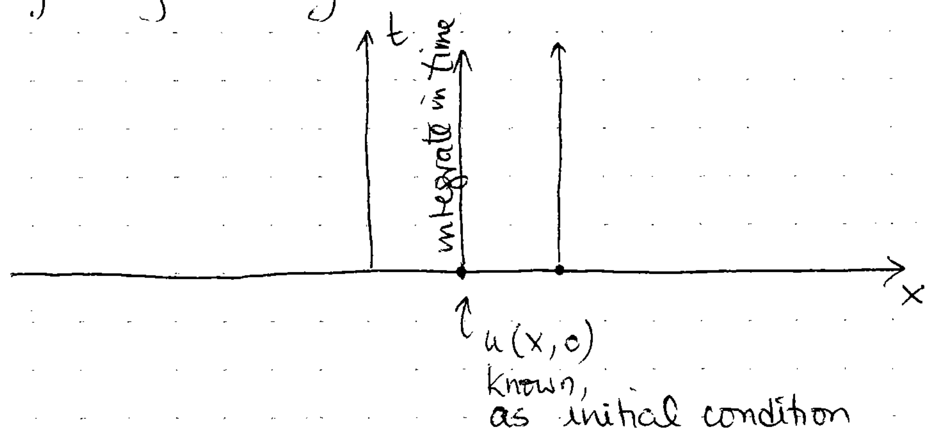
$$= \frac{\partial}{\partial t}(\mu u)$$

$\longrightarrow$ take $\mu = e^{-C_0 t}$ so

$$\frac{\partial}{\partial t}\left(e^{-C_0 t} u\right) = e^{-C_0 t} G(x,t)$$

$$e^{-C_0 t} u(x,t) - e^{-C_0 \cdot 0} u(x,0) = \int_{t'=0}^{t'=t} e^{-C_0 t'} G(x,t')\,dt'$$

Again, this can be done for each value of x separately: we are solving the equation by integrating along lines of constant x



<u>Initial conditions (u is known at t=0)</u>

Suppose we require that $u(x,0) = 3x$ then

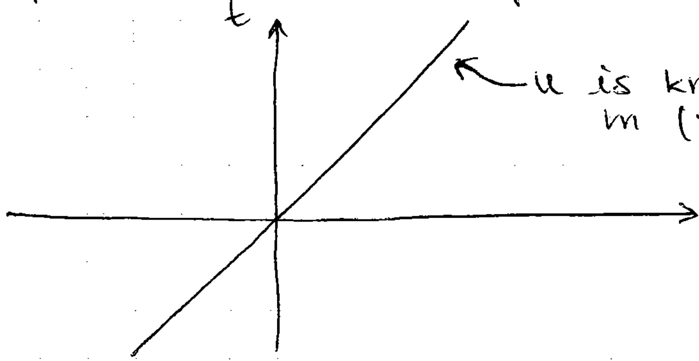
$$u(x,t) = e^{+C_0 t} u(x,0) + \int_{t'=0}^{t'=t} e^{-C_0(t'-t)} G(x,t')\,dt'$$

$$= 3x e^{+C_0 t} + \int_{t'=0}^{t'=t} e^{-C_0(t'-t)} G(x,t')\,dt'$$

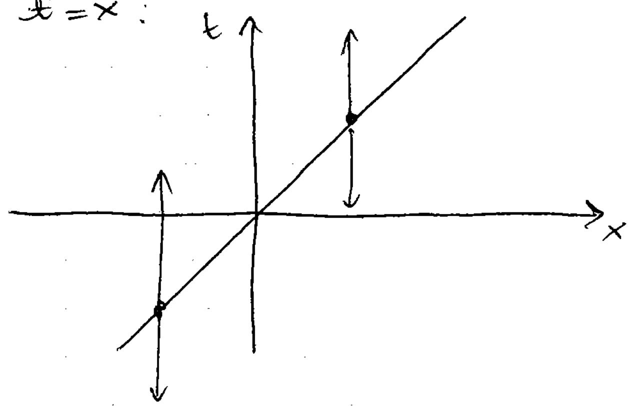
$\rightarrow$ a unique solution.

Other kinds of additional condition

(1) Suppose instead we require that $u(x,x) = 3x$



Then, instead of integrating from $t'=0$, we integrate from $t'=x$:



Mathematically:

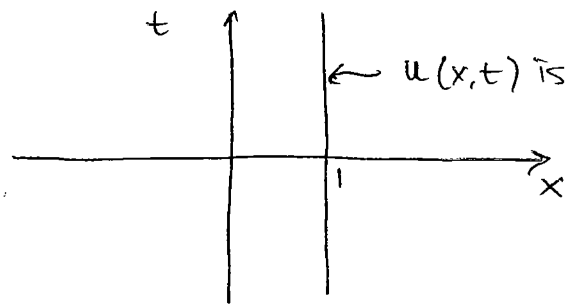
$$e^{-c_0 t} u(x,t) - e^{-c_0 x} u(x,x) = \int_{t'=x}^{t'=t} e^{-c_0 t'} G(x,t')\, dt'$$

$$\Rightarrow \quad e^{-c_0 t} u(x,t) = e^{-c_0 x} \cdot 3x + \int_x^t e^{-c_0 t'} G(x,t')\, dt'$$

$$u(x,t) = e^{-c_0 (x-t)} \cdot 3x + \int_x^t e^{-c_0 (t'-t)} G(x,t')\, dt'$$

$\rightarrow$ again, there is a unique solution to the PDE with the given additional condition.

(2) Now suppose we set $G=0$ and try to impose as additional condition $u(1,t) = 2t$



Problem! The additional condition doesn't satisfy the equation

$$\frac{\partial u}{\partial t} = 2 \quad \rightarrow \quad u_t - G u = 2 - 2Gt \neq 0 .$$

$\rightarrow$ there are no solutions to the equation!

(3) Now suppose $u(1,t) = 2e^{Gt}$ then

$$u_t - G u = 2Ge^{Gt} - 2Ge^{Gt} = 0 \quad ✓$$

$\Rightarrow$ the additional condition satisfies the equation

But note that any function of the form
$$u(x,t) = f(x)\, e^{Gt}$$
satisfies the PDE and the additional condition provided $f(1) = 2$

$\Rightarrow$ there are an ∞ of solutions to the problem!

Conclusion: • Depending on the additional conditions chosen, there can be one, no or an ∞ of solutions to the problem. Case (1) is well-posed while cases (2) and (3) are ill-posed

• What is the difference between cases (1), (2) and (3)?

Note that in case (1), the additional condition crosses all lines of constant x, while in cases (2) and (3), the additional condition is a line of constant x.

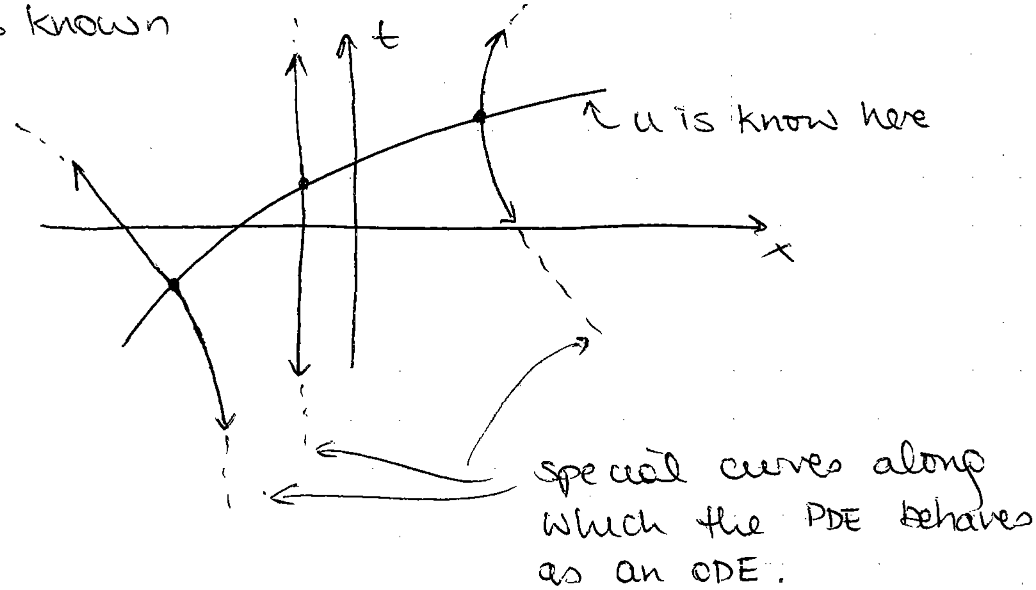
2.2.2 Gearing up to the general method

Now consider the linear transport equation with constant coefficients.

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = c_1 u + c_0$$

where a, b, c_1, c_0 are constants.

<u>Idea</u>: We would like to find curves (as before) along which we could integrate the PDE <u>as if</u> it were an ODE, from an initial or additional condition line where $u(x,t)$ is known



<u>Review of parametric curves</u>

Any curve in $\mathbb{R}^n$ can be represented by a set of parametric equations

$$\begin{cases} x_1 = f_1(s) \\ x_2 = f_2(s) \\ \vdots \\ x_n = f_n(s) \end{cases}$$

where s is the parameter.

<u>Examples:</u> A circle in $\mathbb{R}^2$ (x,y) centred on $(0,0)$ ha[s] the equation

$$\begin{cases} x = R\cos(s) \\ y = R\sin(s) \end{cases}$$

where R is the radius

• A straight line in $\mathbb{R}^2$ has the parametric equation

$$\begin{cases} x = as + c \\ y = bs + d \end{cases}$$

Check : eliminate s to get

$$y = b\left(\frac{x-c}{a}\right) + d = \frac{b}{a}x + \left(d - \frac{bc}{a}\right)$$

Property of parametric curves

The tangent vector to the curve $\{f_1(s) \dots f_n(s)\}$ is

$$\underline{df} = \begin{pmatrix} df_1/ds \\ df_2/ds \\ \vdots \\ df_n/ds \end{pmatrix}$$

Examples: • the tangent vector to the line

$$\begin{cases} x = as + c \\ y = bs + d \end{cases} \quad \text{is} \quad \underline{df} = \begin{pmatrix} dx/ds \\ dy/ds \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

• Suppose you are travelling from SC to Big Sur. Your trajectory is given by the parametric curve

$$\begin{pmatrix} x(t) \\ y(t) \\ h(t) \end{pmatrix} \begin{array}{l} \leftarrow \text{latitudinal position } x \\ \leftarrow \text{longitudinal position } y \\ \leftarrow \text{height} \end{array}$$

Your velocity is the tangent vector to the trajectory

$$\underline{v} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dh}{dt} \end{pmatrix} \begin{array}{l} \leftarrow \text{North-South velocity} \\ \leftarrow \text{East-West velocity} \\ \leftarrow \text{vertical velocity.} \end{array}$$

Note: A parametrization is NOT unique

Example : $\begin{cases} x = R\sin s \\ y = R\cos s \end{cases}$ and $\begin{cases} x = R\sin(s^2) \\ y = R\cos(s^2) \end{cases}$ represent the same curve

Back to the first order PDE

Step 1: We represent the additional condition curve as a parametric curve with parameter s

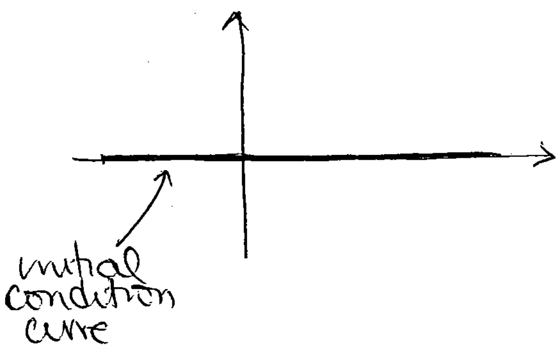
Suppose we know $u(x,t)$ on a particular curve Γ in the (x,t) plane. Let's parametrize Γ with the functions $x_0(s)$, $t_0(s)$ such that

$$\Gamma = \begin{cases} x_0(s) \\ t_0(s) \end{cases}$$

then on this curve $u(x_0(s), t_0(s)) = u_0(s)$

Examples: • Suppose we want to impose

$$u(x, 0) = 3x$$



The initial condition curve has $t=0$ for all x $\rightarrow$

Parametrize it (for example) as

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s$$

or we could also use

$$\begin{cases} x_0(s) = s^2 \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s^2$$

• Suppose $u(x, x^2) = e^x$

then

$$\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases} \Rightarrow u_0(s) = e^s$$

Note: Since there are many possible parametric representations of the same curve, always try to choose the simplest one:

Prefer $\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases}$ over $\begin{cases} x_0(s) = \ln(s^2) \\ t_0(s) = [\ln(s^2)]^2 \end{cases}$!