PDE Course AMS 211

# CHAPTER 1 Introduction

## 1.1 Definitions

- A PDE describes a relation between a function and its partial derivatives.

  Let $u$ be a scalar function in $\mathbb{R}^n$ then the general form of a PDE is

$$\mathcal{F}\Big(\underbrace{x_1,\dots,x_n}_{\text{the independent variables}},\ u,\ \underbrace{\frac{\partial u}{\partial x_1},\dots\frac{\partial u}{\partial x_n},\ \frac{\partial^2 u}{\partial x_1^2},\ \frac{\partial^2 u}{\partial x_1\partial x_2}\ \cdots}_{\text{the partial derivatives.}}\Big)=0$$

a functional of

the dependent variable

- The order of a PDE is defined to be the order of the highest derivative in the equation

- Linear vs nonlinear: a PDE is called a linear equation if $\mathcal{F}$ is a linear combination of $u$ and all its derivatives.

$$\mathcal{F}\left(x_1,\dots x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\cdots\frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1}\cdots\right) = 0$$

$$\Leftrightarrow\ b(x_1,\dots x_n) = a_0(x_1\dots x_n)u + a_1(x_1\dots x_n)\frac{\partial u}{\partial x_1} + a_2(x_1\dots x_n)\frac{\partial u}{\partial x_2} + \dots a_n(x_1,\dots x_n)\frac{\partial u}{\partial x_n}$$

$$+\ a_{2,1}(x_1\dots x_n)\frac{\partial^2 u}{\partial x_1} + a_{2,2}(x_1\dots x_n)\frac{\partial^2 u}{\partial x_1\partial x_2} + \dots \quad \text{etc}$$

If $b(x_1,\dots x_n)=0$ then the PDE is homogeneous

A nonlinear function is not linear.

Examples $u_x = 7$ is a linear non-homogeneous PDE

$u_x + y u_y = 7y$ ———————————

$u_t + 3xu + u_x = 0$ is a linear homogeneous PDE

$u^2 + 3u_x + 2u_y = 0$ is a nonlinear PDE

Amongst nonlinear equations we distinguish

- semi-linear equations ; an equation is semilinear if the term involving the highest derivative is linear (i.e, it only depends on the independent variable)

  example $u_{tt} + u^3 u_x = 3x + y$ is semilinear

- quasilinear equations, an equation is quasilinear (but not semilinear) if the term involving the highest derivative depends only on the independent variable and on $u$.

  example $u\, u_{tt} + u^3 u_x = 3x + y$ is quasilinear but not semilinear

  $e^u u_{xx} + u_{xxxx} + u_t = 0$ is actually semilinear but

  $e^u u_{xxxx} + u_{xx} + u_t = 0$ is quasilinear

- There exist systems of PDES (in which case $u$ becomes a vector of functions, and $\{F_i\}$ a set of functionals:

$$F_i\left(x_1 \dots x_n, u_1 \dots u_m, \frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1} \dots \frac{\partial u_m}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \dots \frac{\partial u_m}{\partial x_2}, \dots \text{etc}\right) \quad \text{for } i = 1 \to m$$

is an $m$-dimensional system ($m$ equations for $m$ variables).

Example: A classical source of systems of PDEs comes from the description of fluid motions:

Suppose we study the velocity field $\underset{\sim}{u} = (u_x, u_y, u_z)$

$u_x$ = velocity component in x- direction

$u_y$ = ——————— y- ———

$u_z$ = ——————— z- direction

Then the Navier-Stokes equation(s) describes the evolution of this fluid: in cartesian coordinates:

$$
\begin{cases}
\dfrac{\partial u_x}{\partial t} + u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} + u_z \dfrac{\partial u_x}{\partial z} = -\dfrac{\partial p}{\partial x} - \nu \nabla^2 u_x \\
\dfrac{\partial u_y}{\partial t} + u_x \dfrac{\partial}{\partial x} u_y + u_y \dfrac{\partial}{\partial y} u_y + u_z \dfrac{\partial}{\partial z} u_y = -\dfrac{\partial p}{\partial y} - \nu \nabla^2 u_y \\
\dfrac{\partial u_z}{\partial t} + u_x \dfrac{\partial}{\partial x} u_z + u_y \dfrac{\partial}{\partial y} u_z + u_z \dfrac{\partial}{\partial z} u_z = -\dfrac{\partial p}{\partial z} = -\nu \nabla^2 u_z \\
\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} + \dfrac{\partial u_z}{\partial z} = 0
\end{cases}
$$

or in short:

$$
\begin{cases}
\dfrac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla p + \nu \nabla^2 \underline{u} \\
\nabla \cdot \underline{u} = 0
\end{cases}
$$

$\Rightarrow$ Most generally, PDEs arising from physical (i.e real) systems are in vector form rather than in a coordinate-dependent form

This is because the solution to the PDE should NOT depend on the coordinate system chosen to represent the physical system

= Principle of COVARIANCE

## 1.2 Examples of PDEs

PDEs are often derived from physical problem. Even when not, they may look like PDEs that are, and understanding intuitively how the physical problem behaves says a lot about the solution

### 1.2.1 Transport equation (simple case)

Idea: given a concentration/density of some quantity, and a velocity field, how does the concentration/density change as a result of advection by the velocity field?

Equation
$$\frac{\partial c}{\partial t} + \nabla\cdot\left(c\underset{\sim}{v}\right) = \underbrace{F(\underset{\sim}{x})}_{\text{source/sink term}}$$

with initial condition
$$c(\underset{\sim}{x}, t=0) = c_0(\underset{\sim}{x})$$

$\underset{\sim}{v}$ = velocity field (known)

$\underset{\sim}{x}$ = position vector

$c$ = concentration (unknown).

Example of use:
- tracking CFCs in atmosphere
- dye in fluid
- heat convection

⋮

Where does it come from? Integrate over volume $V$

$$\int_V \left(\frac{\partial c}{\partial t} + \nabla\cdot(c\underset{\sim}{v})\right) dV = \int_V F(\underset{\sim}{x})\, dV$$

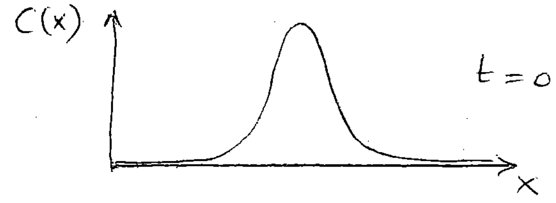
use divergence theorem:

$$\int_V \nabla\cdot\underset{\sim}{A}\, dV = \int_{\substack{\text{surface}\\ \text{around } V}} \underset{\sim}{A}\cdot \underset{\sim}{dS}$$

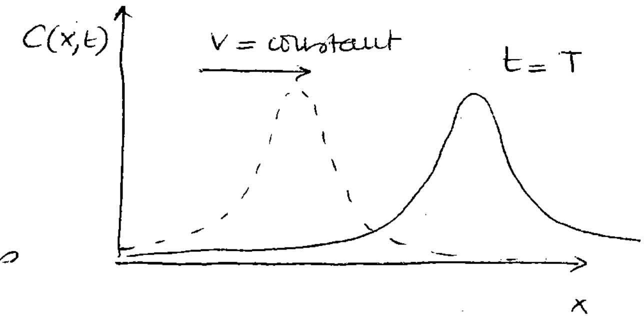
← normal to surface

So
$$\underbrace{\frac{\partial}{\partial t}\int_V c\, dV}_{\substack{\text{total change}\\ \text{of amount of } c\\ \text{in volume}}} + \underbrace{\int_{\substack{\text{surface}\\ \text{around } V}} c\underset{\sim}{v}\cdot\underset{\sim}{ds}}_{\substack{\text{flux of } c \text{ through}\\ \text{the surface}}} = \underbrace{\int_V F(\underset{\sim}{x})\, dV}_{\substack{\text{total source/sink}\\ \text{within volume}}}$$

**Example** • Basic 1D example; with constant velocity field




this is advected by a constant velocity field: what happens?




$$\frac{\partial c}{\partial t} + v_0 \frac{\partial c}{\partial x} = 0$$

→ a linear, first order PDE, homogeneous with constant coefficients

(see PPT)

• Can add a source term : $\frac{\partial c}{\partial t} + v_0 \frac{\partial c}{\partial x} = A e^{-x^2}$

→ a linear, first order, inhomogeneous PDE ↑ example

• Or, can have a velocity dependent on x

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(V C) = 0 \Rightarrow \frac{\partial c}{\partial t} + C\frac{\partial V}{\partial x} + V\frac{\partial c}{\partial x} = 0$$

→ a linear PDE still

• Example in 2D

this time $V = \begin{pmatrix} V_1(x,y) \\ V_2(x,y) \end{pmatrix}$

and
$$\frac{\partial c}{\partial t} + \nabla\cdot(\underline{V}C) = \frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(V_1(x,y)C\right) + \frac{\partial}{\partial y}\left(V_2(x,y)C\right)$$
$$= \frac{\partial c}{\partial t} + V_1\frac{\partial c}{\partial x} + V_2\frac{\partial c}{\partial y} + C\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y}\right) = 0$$

Note that we can rewrite it as

$$\frac{\partial c}{\partial t} + V_1\frac{\partial c}{\partial x} + V_2\frac{\partial c}{\partial y} = -C\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y}\right) = -C\nabla\cdot V$$

↑ looks like a source term!

The $\nabla\cdot\underline{v}$ term represents compression/expansion resulting from the velocity field

- <u>Important point</u> PDEs resulting from physical problems should <u>NOT</u> depend on the coordinate system used (principle of covariance)

so

$$\frac{\partial c}{\partial t} + \nabla\cdot(c\underset{\sim}{v}) = \frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(cv_1) + \frac{\partial}{\partial y}(cv_2)$$

Say $\underset{\sim}{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$ then $= \frac{\partial c}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rcv_a) + \frac{1}{r}\frac{\partial}{\partial \theta}(cv_b) = 0$

in cartesian system $(x,y)$ ; in cylindrical system $(r,\theta)$

Both equations must represent the <u>same</u> system → yield the <u>same</u> solution

<u>Conclusion</u>: always choose the coordinate system that yields the <u>simplest</u> form of the equation <u>AND</u> its boundary/initial conditions.

## 1.2.2 Burger's Equation

- A "transport" equation for the velocity field itself!

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

→ a quasilinear first order equation (homogeneous)

<u>Note</u>: it can be rewritten as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[F(u)\right] = 0 \quad \text{where} \quad F(u) = \frac{u^2}{2}$$

## 1.2.3 Conservation laws

$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[F(u)\right] = 0$ can be generalized in many-D

to $\frac{\partial u}{\partial t} + \nabla\cdot F = 0$, which is called a conservation law

In its integral formulation, we see $F$ = flux of $u$ through any surface (infinitesimal).

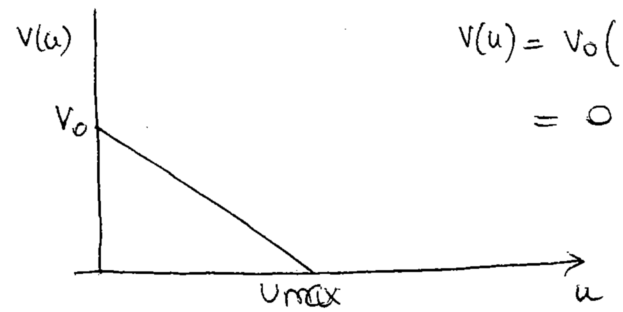
If $F = F(u)$ it is a <u>conservative system</u>

example: Traffic flow. (in 1D)

$u$ = number of cars on a wide motorway
(imagine cars can be side by side).

The flux of cars is equal to $u\,v(u)$
where the velocity $v$ depends on $u$
(lots of cars → low speed
many cars → fast speed).

Simplest formulation:




$$V(u) = V_0\left(1 - \frac{u}{U_{max}}\right) \quad \text{if } 0 < u < U_{max}$$
$$= 0 \quad \text{otherwise.}$$

⇒ equation is $\dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left[V_0 u\left(1 - \dfrac{u}{U_{max}}\right)\right] = 0$

or, $\dfrac{\partial u}{\partial t} + \left(V_0\left(1 - \dfrac{u}{U_{max}}\right) - \dfrac{V_0 u}{U_{max}}\right)\dfrac{\partial u}{\partial x} = 0$

another quasilinear equation

## 1.2.4 The heat equation (the diffusion equation)

Many quantities can satisfy a transport equation like

$$\frac{\partial u}{\partial t} + \nabla \cdot \underset{\sim}{F} = 0$$

but depending on the physical problem, $\underset{\sim}{F}$ can take many forms.

example ① in a metal bar, there is no fluid flow but heat is still transported from hot points to cool point

J.B. Fourier postulated that $\underset{\sim}{F} = -k\nabla T$

heat flux ↗ ↑ heat conduction coefficient.

so the heat equation is

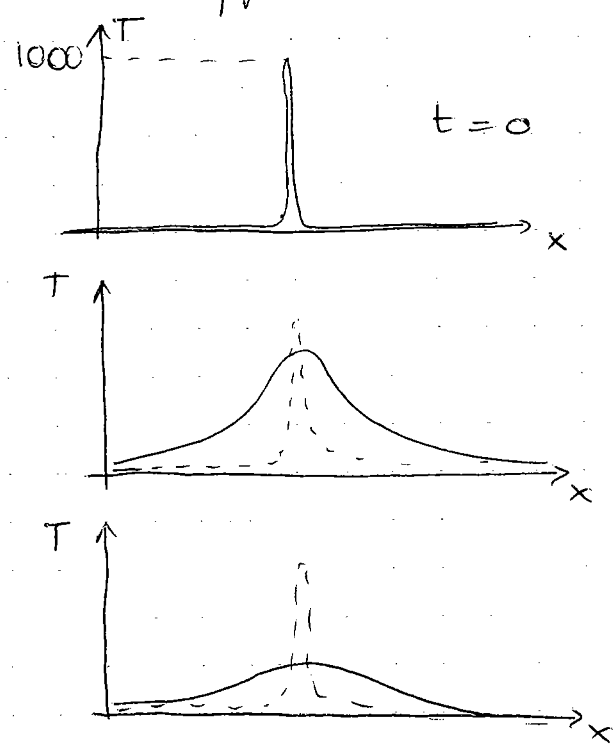
$$\frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T)$$

$\Rightarrow$ in a metal rod (1D) with constant k:

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$ $\leftarrow$ prototype heat equation in 1D

<u>example of solution</u>

at $t=0$, heat up one point in the rod to 1000K.
then let go $\rightarrow$ what happens?




SEE PPT

SEE PPT

<u>Example 2</u> a probabilistic derivation of the diffusion equation (Brownian motion).

Imagine a lattice (in 1D); Define the concentration $C(x,t)$ as the expected number of particles at position $x$, time $t$.

Particles have equal probability to move left or right ($p$) and probability to stay where they are ($1-2p$)

So $$C(x,t+\Delta t) = p\left(C(x-\Delta x,t) + C(x+\Delta x,t)\right) + (1-2p)\,C(x,t)$$

Now assume $\Delta t$ small <u>and</u> $\Delta x$ small then

$$C(x,t) + \Delta t \frac{\partial C}{\partial t} = p\left[2C(x,t) + \Delta x^2 \frac{\partial^2 C}{\partial x^2}\right] + (1-2p)C(x,t)$$
$$= C(x,t) + p\Delta x^2 \frac{\partial^2 C}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial C}{\partial t} = p\frac{\Delta x^2}{\Delta t}\frac{\partial^2 C}{\partial x^2}$$

define this as the diffusion coefficient $K$.

<u>Note 1</u>: In the presence of forces this derivation leads to Fokker–Planck equation.

<u>Note 2</u> From more general considerations, you can derive all the PDEs of fluid mechanics from statistical averaging of ensemble properties of individual particles. <u>Kinetic theory</u> (Boltzmann's equation and its moments).

## 1.2.5 Poisson & Laplace equations

- typically: $\nabla^2 \phi = f(\underset{\sim}{x})$

  (in cartesian $\phi_{xx} + \phi_{yy} = f(x,y)$ for example).

- commonly occurs when studying systems of particles interacting with a force following an inverse square law

  $\left( |\underset{\sim}{F}| \propto \frac{1}{(\text{distance})^2} \right)$

Proof: say $\underline{F}(\underline{r}) = \int \frac{\rho(\underline{r}')}{|r-r'|^3} (\underline{r} - \underline{r}')\, dr'$

then (1) $\nabla \times \underline{F} = 0$ (because the $\nabla \times$ operator acts on $\underline{r}$, so

$$\nabla \times F = \int \nabla \times \left( \frac{\rho(r')}{|r-r'|^3} (\underline{r}-\underline{r}') \right) dr'$$

$$= 0 \quad \text{see handout}$$

(2) $\nabla \cdot F = 4\pi \rho(\underline{r})$ because $\nabla \cdot$ operator acts on $\underline{r}$, so

$$\nabla \cdot F = \int \nabla \cdot \left( \frac{\rho(r')(\underline{r}-\underline{r}')}{|r-r'|^3} \right) dr'$$

see handout $\quad = \int 4\pi \rho(r')\, \delta(\underline{r}-\underline{r}')\, dr'$

so $\nabla \times F = 0 \Rightarrow \underline{F}$ is conservative $\Rightarrow \underline{F} = \nabla \phi$

$\nabla \cdot F = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 4\pi \rho$

- Also occurs in seeking steady-state solutions of the diffusion equation

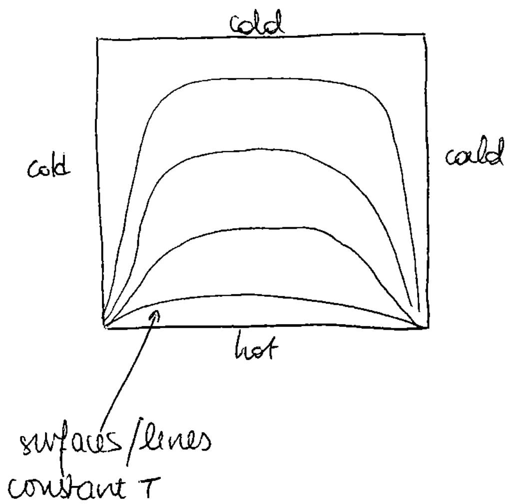
example: What is the temperature profile in a computer chip given that it is constantly heated from 1 side & cooled on the other?

Names: Poisson equation: $\nabla^2 \phi = f(x)$

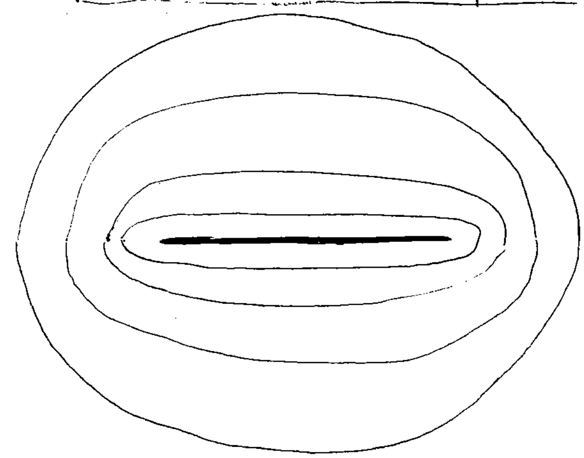
Laplace equation: $\nabla^2 \phi = 0$

- typical solutions

Heated plate
(2D)




Gravitational potential of a spiral galaxy or temperature profile around heated pancake



## 1.2.6 The wave/oscillations equation

- Many physical problems are associated with waves/oscillations, whenever there exists a restoring force (a force that acts against the motion but which doesn't dissipate energy).

  e.g. gravity $\rightarrow$ gravity waves (water, ...)
  tension on a string $\rightarrow$ string oscillations
  pressure $\rightarrow$ sound waves.

- We distinguish between travelling waves and standing waves (oscillations).

- Equation: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

Example: • a travelling pressure wave in a tube (1D)

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}$$

or in space (3D) $\frac{\partial^2 p}{\partial t^2} = c^2\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right)$

• the displacement of an instrument string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

or of a cylindrical drum

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

The proportionality constant, $c$, is the wave <u>speed</u> (velocity of propagation of information)

$c$ doesn't need to be constant

- can depend on space, → waves in an inhomogeneous medium (see light waves air → glass, refraction; or water waves going to shallower water, etc).
- can depend on the amplitude of the wave (nonlinear waves, soliton solutions).

(PPT).

## 1.3 Additional conditions & well-posedness (a first look at)

- When modelling physical problems, the PDE is always accompanied by additional conditions, usually in the form of
  - initial conditions (for a time-dependent problem)
  - boundary conditions (for a problem on a finite domain)
  - regularity conditions (either, regularity/bound at infinity, or regularity at a coordinate singularity)

The behaviour of a solution depends as much of the PDE than on these additional conditions

- For a given PDE with given additional conditions, there can be no, one or many possible solutions

  example. $u_t = u_x$

  - Given this PDE without any additional conditions, there are an infinity of solutions ($u = c$ for all $c \in \mathbb{R}$)
  - Given this PDE together with $u(x, t=0) = \phi(x)$ there is a unique solution (see Chapter 2)
  - Given this PDE with $u(x, t=x) = \phi(x)$ then there is no solution (see Chapter 2) or an infinite # of solutions.

- For a given equation and set of additional conditions, a small change in the ~~state~~ parameters of the equation or of the conditions can lead to a big change in the solution.

Example $\quad u_t = -u_{xx} \qquad t>0$

$u(x,0) = 1$

→ obvious solution is $u(x,t) = 1$
but if we had chosen $u(x,0) = 1 + \frac{1}{n} \cdot \sin(nx)$
then the solution is

$$u(x,t) = 1 + \frac{1}{n} e^{n^2 t} \sin(nx) \qquad \text{(CHECK THIS)}$$

Now for $n$ large enough $1 + \frac{1}{n}\sin(nx) \simeq 1$
but after a time $t$ large enough $\frac{e^{n^2 t}}{n} \gg 1$
so a small difference in initial conditions creates an enormous difference in the final solution.

Definition:
a PDE (or set of PDE) and its associated additional conditions is a well-posed problem if
- it has a solution
- this solution is unique
- the structure of the solution is unchanged by infinitesimal variations of the parameters and/or of the additional conditions.

Typically: • a well-thought, well-modelled physical problem will result in a well-posed problem. because in nature, the solution exists.
(although simplifying assumptions & shortcuts often lead to ill-posed problems)

BUT: not necessarily always the case
- multiple solutions can exist in which slight changes in the boundary conditions lead to one, or the other equilibrium
- sometimes, it is the discontinuous solutions that interest us (@shock physics)
⇒ called weak solutions