## 3. Matrix Notation

The system

$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 0 \\ & & 2y & - & 8z & = & 8 \\ -4x & + & 4y & + & 9z & = & -9 \end{array}$$

has coefficient matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 4 & 9 \end{bmatrix}$

and augmented matrix $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 4 & 9 & -9 \end{array}\right]$.

A **matrix** $A$ with $m$ rows and $n$ columns has **size** $m \times n$.

We write $a_{ij}$ for the entry of matrix $A$ in row $i$ and column $j$.

## 4. Row Echelon Form (REF)

A **leading entry** of a row refers to the left most non-zero entry (in a non-zero row).

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**Definition:** A matrix is in **Echelon Form** or **Row Echelon Form (REF)** if:

1. Any rows consisting of entirely zeros are at the bottom.

2. In each non-zero row, the leading entry of the row is in a column to the left of any leading entries below it.

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Note that this means that all entries in a column below a leading entry are zero.

Echelon forms are often thought of as having a staircase pattern.

Examples:

(i) $\begin{bmatrix} 2 & 1 & 6 & 4 & -2 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & 3 & -3 & 5 & 0 \\ 0 & 1 & 0 & 5 & 2 \\ 0 & 0 & 8 & -4 & 6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 2 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 5 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & -1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 7 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & * \end{bmatrix}$

Where each $*$ can be any value.

(iv) $\begin{bmatrix} 2 & 5 & 3 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

All of the above matrices are in echelon form.

## 5. Reduced Row Echelon Form (RREF)

A matrix is in **Reduced Echelon Form** or **Reduced Row Echelon Form (RREF)** if:

1-2. It is in echelon form, AND

3. The leading entry in each nonzero row is 1.

4. Each leading 1 is the only nonzero entry in its column.

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Examples: (let * be any number)

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \quad \text{RREF}$$

$$\begin{bmatrix} 0 & 1 & ** & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 1 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \quad \begin{array}{c} \text{if } a_{13} = 0 \\ \text{RREF} \\ \\ \text{if } a_{13} \neq 0 \\ \text{not RREF} \end{array}$$

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & -1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \text{ Not RREF (Only REF)}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ RREF}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ RREF}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ Not RREF, Not REF}$$

## 6. Elementary Row Operations

Any matrix can be reduced to a matrix in row echelon form by using certain allowable operations.

These operations are called

**elementary row operations**,

and there are 3 types:

1. Interchange any two rows

   Ex. $R_1 \leftrightarrow R_3$

2. Multiply a row by a non-zero constant

   Ex. $R_2' = 5R_2$

3. Replace a row by the sum of itself and a multiple of another row

   Ex. $R_3' = R_3 + 4R_2$

# 7. Solving Linear Systems of Equations

To solve a system of linear equations, we set up the corresponding augmented matrix and use elementary row operations to reduce the matrix.

We work column by column, from left to right and from top to bottom. The strategy is to create a leading entry in a column, and then use it to create zeros below it.

---

**Detailed Example:**

The system

$$\begin{array}{rcrcrcr} x & + & 2y & & & = & 4 \\ -x & + & 3y & + & 3z & = & -2 \\ & & y & + & z & = & 0 \end{array}$$

has the corresponding augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

We have a "1" in position $a_{11}$.
(We will later see that this is called a pivot.)

We will perform row operations to eliminate the non-zero entries (in the first column) below this pivot.

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\sim \quad R_2' = R_2 + R_1 \quad \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 5 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Our first column now has a pivot and zeros below it.

Next we will try for a leading entry/pivot in position $a_{22}$ and a zero entry below it.

Mathematically, it's nice if this leading entry is a "1".

$$\sim \quad \begin{array}{c} \\ R_3 \\ R_2 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 3 & 2 \end{array}\right]$$

$$\sim \quad R_3' = R_3 + (-5)R_2 \quad \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right] \text{(REF)}$$

Note that each column of the coefficient matrix has a leading entry, and that each leading entry is in a column to the left of the leading entry below it.

There are 2 ways to solve the system.

Method 1: Back substitution

(We do this by returning to a system of equations equivalent to the augmented matrix.)

The corresponding system is

$$\begin{array}{rcrcrcl} x & + & 2y & & & = & 4 \\ & & y & + & z & = & 0 \\ & & & - & 2z & = & 2 \end{array}$$

Then, working from the last equation upwards,

$$-2z = 2 \quad \Longrightarrow \quad z = -1$$

$$y + z = 0 \quad \Longrightarrow \quad y = -z = 1$$

$$x + 2y = 4 \quad \Longrightarrow \quad x = 4 - 2y = 4 - 2 = 2$$

Thus the solution is $(x, y, z) = (2, 1, -1)$.

Method 2: Continue row operations on the matrix to put it in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right]$$

$$\sim_{R_3' = -\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\sim_{R_2' = R_2 + (-1)R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\sim_{R_1' = R_1 + (-2)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

The solution is $(x, y, z) = (2, 1, -1)$.

## 8. Uniqueness of RREF

Note 1: Echelon form of a matrix is **not** unique.

$$\begin{bmatrix} 3 & 1 & 0 & -11 \\ 2 & 1 & -1 & -4 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{ (REF)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{(REF)} \sim \begin{bmatrix} 3 & 3 & 3 & 9 \\ 0 & 2 & 6 & 20 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{(REF)}$$

Note 2: **Reduced** Echelon form of a matrix **is** unique.

$$\begin{bmatrix} 3 & 1 & 0 & -11 \\ 2 & 1 & -1 & -4 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ (RREF)}$$

## 9. Terminology (Over 2 pages!)

A **pivot position** is a position of a leading entry in an echelon form of a matrix.

A **pivot** is a non-zero number that is in a pivot position.

A **pivot column** is a column that contains a pivot position.

---

**Example:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Both $(1,1)$ and $(2,3)$ are pivot positions.

$a_{11} = 1$ and $a_{23} = 4$ are pivots

The first and third columns are pivot columns.

A **basic variable** is a variable that corresponds to a pivot column.

A **free variable** is a variable that is **not** a basic variable, ie, a variable that corresponds to a non-pivot column.

---

**Example:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

If this is the coefficient matrix for a system in unknowns $a, b, c$, then $a$ and $c$ are basic variables since their columns are pivot columns.

Note that $b$ is a free variable, since there is no pivot in the second column.

## 10. Solutions of Linear Systems and Row Equivalence of Matrices

**Example:** Determine whether the following linear system has a solution.

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 3 \\ x & - & y & + & z & = & 1 \\ -2x & - & 4y & - & 2z & = & 4 \end{array}$$

**Solution:** The associated augmented matrix is:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ -2 & -4 & -2 & 4 \end{array}\right]$$

and we perform elementary row operations to put it in row echelon form:

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right]$$

The last row says

$$0 \cdot x + 0 \cdot y + 0 \cdot z = 10,$$

which is impossible.

The system does not have any solutions, ie the system is inconsistent.

**Example:** Solve the following of linear equations

$$\begin{array}{rcrcrcr} x & + & 2y & - & 3z & = & 3 \\ -2x & - & 5y & + & 4z & = & 5 \\ -5x & - & 13y & + & 9z & = & 18 \end{array}$$

**Solution:** We reduce the associated augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 3 \\ -2 & -5 & 4 & 5 \\ -5 & -13 & 9 & 18 \end{array}\right]$$

$$\sim \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 + 5R_1 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & -3 & -6 & 33 \end{array}\right]$$

$$\sim \begin{array}{l} R_2' = -R_2 \\ R_3' = R_3 + 3R_2 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 3 \\ 0 & 1 & 2 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{(REF)}$$

$$\sim \; R_1' = R_1 - 2R_2 \left[\begin{array}{rrr|r} 1 & 0 & -7 & 25 \\ 0 & 1 & 2 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{(RREF)}$$

The third column does not have a pivot position, so the corresponding variable $z$ is a free variable, say $z = t$.

The RREF gives the equations

$$\begin{array}{rcrcrcr} x & & & - & 7z & = & 25 \\ & & y & + & 2z & = & -11 \end{array}$$

which gives

$$\begin{array}{lcccr} x & = & 7z + 25 & = & 7t + 25 \\ y & = & -2z - 11 & = & -2t - 11 \\ z & & & = & t \end{array}$$

The general solution in **parametric vector form** is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7t + 25 \\ -2t - 11 \\ t \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 25 \\ -11 \\ 0 \end{bmatrix}$$

The system is consistent and has infinitely many solutions.

**Exercise:** Find the value of the constant $k$ such that the following system has

(i) no solution
(ii) infinitely many solutions
(iii) unique solution

**Solution:** We reduce the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ -2 & -3 & 2 & -1 \\ -5 & -8 & 5 & k \end{array}\right]$$

$$\begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 + 5R_1 \end{array} \left[\begin{array}{rrr|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 5+k \end{array}\right]$$

$$\begin{array}{l} R_1' = R_1 + (-2)R_2 \\ \\ R_3' = R_3 + (-2)R_2 \end{array} \left[\begin{array}{rrr|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 3+k \end{array}\right]$$

The last line of the REF of the matrix says

$$0 = 3 + k$$

The system will be consistent if $3 + k = 0$ and inconsistent otherwise.

Hence,
(i) no solution if $k \neq -3$
(ii) infinitely many solutions if $k = -3$
(iii) unique solution not possible (no such $k$)

**Example:** Find the value(s) of the constant $h$ and $k$ such that the following system has

(i) no solution
(ii) infinitely many solutions
(iii) unique solution

$$\left[\begin{array}{cc|c} 1 & h & 1 \\ 3 & 3 & k \end{array}\right]$$

**Solution:** Row-Reducing the system gives

$$\underset{R_2' = R_2 + (-2)R_1}{\sim} \left[\begin{array}{cc|c} 1 & h & 1 \\ 0 & 3-3h & k-2 \end{array}\right]$$

(i) no solution if $3-3h=0$ and $k-2 \neq 0$, that is if $h=1$ and $k \neq 2$

(ii) infinitely many solutions if $3-3h=0$ and $k-2=0$, that is if $h=1$ and $k=2$

(iii) unique solution if $3-3h \neq 0$, that is $h \neq 1$ (Note that any value for $k$ is fine here).

---

**Exercise:** Solve the same question with the system

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & -h & k \end{array}\right].$$

**Equivalent Systems of Equations**

The system

$$\begin{aligned} x + 2y - 3z &= 3 \\ -2x - 5y + 4z &= 5 \qquad (*) \\ -5x - 13y + 9z &= 18 \end{aligned}$$

can also be written in the form

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -5 & 4 \\ -5 & -13 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} \qquad (**)$$

and also written in the form

$$x \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + y \begin{bmatrix} 2 \\ -5 \\ -13 \end{bmatrix} + z \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} \qquad (***)$$

Solving the linear system (*) is the same as solving the matrix equation (**) and also the same as solving the vector equation (***).

**Row Equivalence of Matrices**

**Definition:**

The matrix $A$ and the matrix $B$ are **row equivalent** if there is a sequence of elementary row operations that converts $A$ into $B$.

**Theorem:**

The matrix $A$ and the matrix $B$ are row equivalent if and only if they can be reduced to the same row echelon form.

## 11. Rank of a Matrix

**Definition:**

The **rank** of a matrix is the number of nonzero rows in its row echelon form.

(Any row echelon form of a matrix $A$ will have the same number of rows of zeros).

**Theorem: The Rank Theorem**

Let $A$ be the coefficient matrix of a system of linear equations with $n$ variables.

If the system is consistent, then

$$\# \text{ of free variables} = n - \text{rank}(A)$$

**Example 1:**

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

If $A$ is the coefficient matrix of a linear system, then

$$\begin{aligned} n &= \text{3variables (columns)} \\ \text{rank}(A) &= \text{ number of nonzero rows in REF } = 2 \end{aligned}$$

$$\Longrightarrow \text{ \# of free variables} = n - \text{rank}(A) = 3 - 2 = 1$$

**Example 2:**

$$\text{Let } B = \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

If $B$ is the coefficient matrix of a linear system, then

$$\begin{aligned} n &= \text{4 variables/columns} \\ \text{rank}(B) &= \text{ number of nonzero rows in REF } = 3 \end{aligned}$$

$$\Longrightarrow \text{ \# of free variables} = n - \text{rank}(A) = 4 - 3 = 1$$

## 12. Homogeneous Systems and Non-Homogeneous Systems

A **homogeneous** system is one where all the constant terms of the equations are equal to zero.

A homogeneous system is **always consistent** (ie always has a solution)

Example: Solve the homogeneous system

$$\begin{array}{rcrcrcrcr} x & + & 3y & - & 2z & - & w & = & 0 \\ -2x & - & 5y & & & + & 4w & = & 0 \\ x & + & 4y & - & 6z & + & w & = & 0 \end{array}$$

Solution:

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 0 \\ -2 & -5 & 0 & 4 & 0 \\ 1 & 4 & -6 & 1 & 0 \end{array}\right]$$

$$\sim \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 - R_1 \end{array} \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 1 & -4 & 2 & 0 \end{array}\right]$$

$$\sim_{R_3' = R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since $z$ and $w$ are free variables (no pivot in their columns), we write $x$ and $y$ in terms of $z$ and $w$.

From the second row,

$$y - 4z + 2w = 0 \implies y = 4z - 2w$$

From the first row,

$$\begin{aligned} x + 3y - 2z - w = 0 \implies x &= -3y + 2z + w \\ &= -3(4z - 2w) + 2z + w \\ &= -10z + 7w \end{aligned}$$

So

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10z + 7w \\ 4z - 2w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

**Theorem:**

If $[A|\vec{0}]$ is a homogeneous system of $m$ linear equations in $n$ unknowns, where $m < n$, then the system has infinitely many solutions.

In the example above, we had a homogeneous system in 3 equations in 4 unknowns, so (by the theorem) we have infinitely many solutions.

Example: Now solve the non-homogeneous system (Note similarity to previous example!)

$$\begin{array}{rcrcrcrcr} x & + & 3y & - & 2z & - & w & = & 2 \\ -2x & - & 5y & & & + & 4w & = & 3 \\ x & + & 4y & - & 6z & + & w & = & 9 \end{array}$$

Solution: We perform the same row operations on the new augmented system to get

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 2 \\ -2 & -5 & 0 & 4 & 3 \\ 1 & 4 & -6 & 1 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 2 \\ 0 & 1 & -4 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So second row says

$$y - 4z + 2w = 7 \implies y = 4z - 2w + 7$$

and first row says

$$x + 3y - 2z - w = 2$$

$$\begin{aligned}\implies x &= -3y + 2z + w + 2\\ &= -3(4z - 2w + 7) + 2z + w + 2\\ &= -10z + 7w - 19\end{aligned}$$

Hence

$$\begin{aligned}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} &= \begin{bmatrix} -10z + 7w - 19 \\ 4z - 2w + 7 \\ z \\ w \end{bmatrix}\\ &= z \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -19 \\ 7 \\ 0 \\ 0 \end{bmatrix}\end{aligned}$$

**Example:** Describe all solutions of the homogeneous equation $A\vec{x} = \vec{0}$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 6 & 0 & 8 & -1 & -2 \\ 0 & 0 & 1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Solution:** The augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 6 & 0 & 8 & -1 & -2 & 0 \\ 0 & 0 & 1 & -3 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot columns: 1,3,6
$\Longrightarrow$ Basic variables: $x_1$, $x_3$, $x_6$

Non-pivot columns: 2,4,5
$\Longrightarrow$ Free varibales: $x_2$, $x_4$, $x_5$

Let $x_2 = r$, $x_4 = s$, $x_5 = t$

From the third row of the augmented matrix, $x_6 = 0$.

From the second row we have

$$x_3 - 3x_4 + 4x_5 + 6x_6 = 0$$

$$\begin{aligned} \Longrightarrow \; x_3 &= 3x_4 - 4x_5 - 6x_6 \\ &= 3s - 4t \end{aligned}$$

From the first row we have

$$x_1 + 6x_2 + 8x_4 - x_5 - 2x_6 = 0$$

$$\begin{aligned} \Longrightarrow \; x_1 &= -6x_2 - 8x_4 + x_5 + 2x_6 \\ &= -6r - 8s + t \end{aligned}$$

(Full solution in parametric vector form on next slide)

The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -6r - 8s + t \\ r \\ 3s - 4t \\ s \\ t \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6r \\ r \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -8s \\ 0 \\ 3s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ -4t \\ 0 \\ t \\ 0 \end{bmatrix}$$

$$= r \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -8 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

where $r, s, t$ are any real numbers.