York University
Faculty of Arts, Faculty of Science and Engineering

Final Examination

April 17, 2008
Mathematics 1505.06
Mathematics for Life and Social Sciences

NAME (print):______________________
(Family) (Given)

SIGNATURE:______________________

STUDENT NUMBER:______________________

Section A. MWF @ 8:30 (CLH A) - Prof. Chawl
Section B. MWF @ 9:30 (CLH A) - Prof. Pietrowsk
Section C. MWF @ 9:30 (VH C) - Prof. Grigull

Section D, T @ 7 (CLH G) - Prof. Mohammed

Section E. MWF @ 10:30 (CLH E) - Prof. Chawla
Section G, MWF @ 10:30 (CLH K) - Prof. Raguimov

Section ☐

Instructions:

1. You have to answer all questions and show all your work.
2. Put answers and rough work on the question paper, using the back pages if necessary.
3. Nonprogrammable and nongraphing calculators are allowed.
4. Exam is for 3 hours.
5. This exam has 18 questions. Make sure you have everything.
6. Total number of mark is 200.

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1. (8 marks)
(a) Solve $|3x+5| < 1$. Express your answer in interval notation.

$-1 < 3x+5 < 1$

$\Rightarrow -6 < 3x < -4$

$\Rightarrow -2 < x < -\frac{4}{3}$.

Ans : $(-2, -\frac{4}{3})$.

(b)Find the largest possible domain of the function $f(x) = \sqrt{2-3x}$. Express your answer in interval notation.

$f(x)$ is defined only when $2-3x \geq 0$.

$\Rightarrow 2 \geq 3x$

$\Rightarrow \frac{2}{3} \geq x$.

Ans : $(-\infty, \frac{2}{3}]$

continues . . .

2. (8 marks)
(a) Solve $ln(4x - 2x^2) - ln(x) = ln(2x - 1)$.

$$\ln(4x - 2x^2) = \ln x + \ln(2x-1)$$

$$\Rightarrow \ln(4x - 2x^2) = \ln x(2x-1)$$

$$\Rightarrow 4x - 2x^2 = x(2x-1) = 2x^2 - x$$

$$\Rightarrow 4x^2 - 5x = 0$$

$$\Rightarrow x(4x-5) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \frac{5}{4} \qquad \therefore x = \frac{5}{4} //$$

↑ but ln(0) is undefined

(b) Find all values of α in the interval $[0, 2\pi)$ that satisfy the equation $2sin\alpha cos\alpha = cos\alpha$.

$$2\sin\alpha\cos\alpha - \cos\alpha = 0$$

$$\Rightarrow \cos\alpha(2\sin\alpha - 1) = 0$$

$$\Rightarrow \cos\alpha = 0 \quad \text{or} \quad \sin\alpha = \frac{1}{2}$$

$$\Rightarrow \alpha = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \alpha = \frac{\pi}{6}, \frac{5\pi}{6}$$

continues ...

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3. (24 marks) Determine the following limits:

(a) $\lim_{x \to \infty} \frac{2e^x + \boxed{3e^{-x}} \to 0}{5e^x - \boxed{7e^{-x}} \to 0}$

$$= \lim_{x \to \infty} \frac{2e^x}{5e^x}$$

$$= \frac{2}{5}$$

(b) $\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x^2 + 5x - 3}$

$$= \lim_{x \to \frac{1}{2}} \frac{(2x-1)(2x+1)}{(2x-1)(x+3)}$$

$$= \lim_{x \to \frac{1}{2}} \frac{(2x+1)}{x+3}$$

$$= \frac{2(\frac{1}{2})+1}{\frac{1}{2}+3}$$

$$= \frac{4}{7}$$

continues ...

(c) $\lim_{x\to\infty}\left(\sqrt{2x^2-1}-\sqrt{2x^2-3x}\right)$

$$= \lim_{x\to\infty}\left(\sqrt{2x^2-1}-\sqrt{2x^2-3x}\right)\cdot\frac{\sqrt{2x^2-1}+\sqrt{2x^2-3x}}{\sqrt{2x^2-1}+\sqrt{2x^2-3x}}$$

$$= \lim_{x\to\infty}\frac{(2x^2-1)-(2x^2-3x)}{\sqrt{2x^2-1}+\sqrt{2x^2-3x}}$$

$$= \lim_{x\to\infty}\frac{3x-1}{\sqrt{2x^2-1}+\sqrt{2x^2-3x}}\cdot\frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}}$$

$$= \lim_{x\to\infty}\frac{3-\frac{1}{x}\to 0}{\sqrt{2-\frac{1}{x^2}}\to 0+\sqrt{2-\frac{3}{x}}\to 0}.$$

$$= \frac{3}{\sqrt{2}+\sqrt{2}} = \frac{3}{2\sqrt{2}}$$

(d) $\lim_{x\to\infty}\left(\frac{\frac{3}{x^2}-\frac{2}{x}}{\frac{2}{x^2}+\frac{7}{x}}\right)$

$$= \lim_{x\to\infty}\frac{\frac{3-2x}{x^2}}{\frac{2+7x}{x^2}}$$

$$= \lim_{x\to\infty}\frac{3-2x}{2+7x}$$

$$= \frac{-2}{7}.$$

continues . . .

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4. (8 marks) A toxin is introduced into a bacterial colony, and t hours later, the population is given by $N(t) = 10000(8+t)e^{-0.1t}$

(a) What was the population when the toxin was introduced?
(b) When is the population maximized? Justify your answer.
(c) Find the maximum population.

a) $N(0) = 10000(8+0)e^{(-0.1)0} = 80000$

b) $N'(t) = 10000e^{-0.1t} + 10000(8+t)(-0.1)e^{-0.1t}$

$= (2000-1000t)e^{-0.1t}$

$N'(t) = 0 \Rightarrow 2000-1000t = 0.$

$\Rightarrow t = 2$

		2	
$N'(t)$	+		−
$N(t)$	increasing		decreasing.

∴ N(t) is maximized at t=2 and.

c) $N(2) = 100000e^{-0.2} \approx 81873.$

continues . . .

5. (18 marks) Given a function $y = f(x) = \frac{x-1}{x^2}$.

(a) Determine intervals on which the function is strictly increasing, strictly decreasing. Find the coordinates of all local (relative) maximum and minimum points (if any).

$$f'(x) = \frac{x^2-(x-1)(2x)}{(x^2)^2} = \frac{-x+2}{x^3}$$

$f'(x) = 0 \Rightarrow x = 2$ ↙ local max $f(2) = \frac{2-1}{2^2} = \frac{1}{4}$

0 2

$f'(x)$	$-$	$+$	$-$
$f(x)$	decreasing	increasing	decreasing.

Ans: $f(x)$ is decreasing on $(-\infty, 0)$ and $(2, \infty)$ and increasing on $(0, 2)$.

It has local maximum at $(2, \frac{1}{4})$ and no local minimum

(b) Determine intervals on which the function is concave up, concave down. Find the coordinates of all inflection points (if any).

$$f''(x) = \frac{x^3(-1)-(2-x)(3x^2)}{(x^3)^2} = \frac{2(x-3)}{x^4}$$

$f''(x) = 0 \Rightarrow x = 3$ ↙ point of inflection $f(3) = \frac{2}{9}$

0 3

$f''(x)$	$-$	$-$	$+$
$f(x)$	Concave down	Concave down	Concave up

Answer: $f(x)$ is concave down on $(-\infty, 0)$ and $(0, 3)$, it is concave up on $(3, \infty)$

It has a point of inflection at $(3, \frac{2}{9})$

continues ...

(c) Sketch the graph of the function.

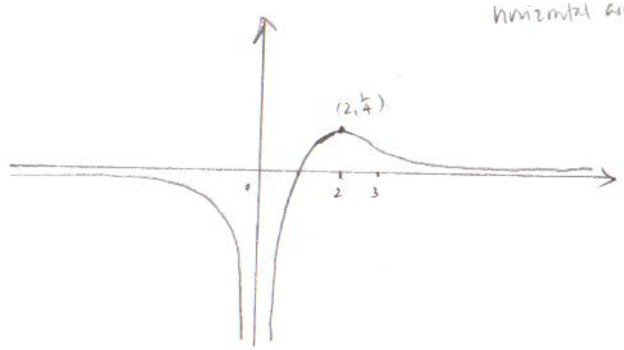
$f(x) = 0 \Rightarrow x = 1$

$\lim_{x \to 0^-} f(x) = -\infty$ $\lim_{x \to 0^+} f(x) = -\infty$

vertical asymptote : $x = 0$.

$\lim_{x \to \infty} f(x) = 0 = \lim_{x \to -\infty} f(x)$

horizontal asymptote : $y = 0$



→ 6. (5 marks) Find a function F(x) such that $F'(x) = \frac{1}{x}$ and F(2)=0.

Antiderivative of $\frac{1}{x}$ is $\ln|x|$

$F(x) = \ln|x| + C$

$F(2) = \ln|2| + C = 0 \Rightarrow C = -\ln 2$

$\therefore F(x) = \ln|x| - \ln 2$.

continues ...

7. (30 marks) Calculate the derivatives for the following functions:
(a) $8e^{x^2}$

$$\frac{d}{dx} 8e^{x^2} = 8 \cdot (2x) e^{x^2} = 16 x e^{x^2}$$

(b) $\frac{(1+3x)^2}{(1+2x)^3}$

$$\frac{d}{dx} \frac{(1+3x)^2}{(1+2x)^3} = \frac{(1+2x)^3 \cdot 2(1+3x) \cdot 3 - (1+3x)^2 3(1+2x)^2 \cdot 2}{[(1+2x)^3]^2}$$

$-b$(c) $ln(x) \cdot cos(a + bx)$, where a and b are constants.

$$\frac{d}{dx}\left(\ln x \cos(a+bx)\right) = \frac{1}{x} \cos(a+bx) + \ln x \cdot \left(-\sin(a+bx)\right) \cdot b$$

$$= \frac{\cos(a+bx)}{x} - b \sin(a+bx) \ln x$$

continues . . .

(d) $2^{x/2}$ $= \left(e^{(\ln 2)}\right)^{\frac{x}{2}} = e^{\frac{\ln 2}{2} x}$

$$\frac{d}{dx} 2^{x/2} = \frac{\ln 2}{2} e^{\frac{\ln 2}{2} x} = \frac{\ln 2}{2} 2^{x/2}$$

(e) Let $f(x) = 2x + e^{2x}$. Find $\frac{d}{dx}(f^{-1}(4 + e^4))$. Note that $f(\underset{y}{2}) = \underset{x}{4 + e^4}$.

$y = f^{-1}(x) \Rightarrow f(y) = x$

$\Rightarrow 2y + e^{2y} = x$

$\Rightarrow 2\frac{dy}{dx} + 2e^{2y}\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}f^{-1}(x) = \frac{1}{2+2e^{2y}}$

$$\frac{d}{dx} f^{-1}(4+e^4) = \frac{d}{dx} f^{-1}(x) \Big|_{x=4+e^4} = \frac{1}{2+2e^{2 \cdot 2}} = \frac{1}{2+2e^4}$$

8. (5 marks) Find the slope of the tangent line to the graph of the following function and determine its equation in the slope-intercept form: $f(x) = 3x - x^2$ at (-2, -10).

$f'(x) = 3 - 2x$

$f'(-2) = 3 - 2(-2) = 7$

tangent line : $y - (-10) = 7(x - (-2))$

$\Rightarrow y + 10 = 7x + 14$

$\Rightarrow y = 7x + 4$

continues . . .

9. (5 marks) Find the global (absolute) extrema of the function: $\frac{lnt}{t}$. State whether it is a global maximum or a global minimum and justify your answer.

$f(t) = \frac{\ln t}{t}$

$f'(t) = \frac{t \cdot \frac{1}{t} - \ln t \cdot 1}{t^2} = \frac{1 - \ln t}{t^2} = 0 \Rightarrow t = e$.

	0		e ↙ local max	
$f'(t)$	not defined	+		−
$f(t)$		increasing		decreasing

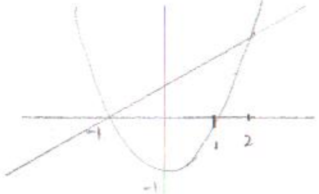
$\lim_{t \to 0} \frac{\ln(t)}{t} = -\infty \Rightarrow f(t)$ has no global min //

Since f(t) is decreasing.
for all t > e
f(t) has a global maximum
at $(e, \frac{1}{e})$ //

10. (5 marks) Find $\frac{dy}{dx}$ where $y = \int_1^x ln(1+t^2)dt$.

$\frac{dy}{dx} = \ln(1+x^2)$

11. (6 marks) Find the area of the region bounded by the line $y = x + 1$ and curve $y = x^2 - 1$.



Find intersection: $x + 1 = x^2 - 1$
$\Rightarrow x^2 - x - 2 = 0$
$\Rightarrow (x-2)(x+1) = 0$
$\Rightarrow x = 2,\ x = -1$

$$
\begin{aligned}
\text{Area} &= \int_{-1}^{2} (x+1) - (x^2-1)\, dx \\
&= \int_{-1}^{2} x - x^2 + 2\, dx \\
&= \left(\frac{x^2}{2} - \frac{1}{3}x^3 + 2x\right)_{-1}^{2} \\
&= \left(\frac{4}{2} - \frac{8}{3} + 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right) \\
&= \frac{9}{2}.
\end{aligned}
$$

continues . . .

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12. (30 marks) Evaluate the following integrals:

(a) $\int \frac{(lnx)^2}{x} dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{1}{3}u^3 + C$
$= \frac{1}{3}(\ln x)^3 + C$

(b) $\int_0^{\pi/4} tanxsec^2x\ dx$

Let $u = \tan x \Rightarrow du = \sec^2 x\, dx$
when $x = 0,\ u = 0$
$x = \frac{\pi}{4},\ u = 1$

$\int_0^{\pi/4} \tan x \sec^2 x\, dx = \int_0^1 u\, du = \frac{1}{2}u^2\Big]_0^1 = \frac{1}{2}$.

Solution ②

Let $u = \sec x \Rightarrow du = \sec x \tan x\, dx$
when $x = 0,\ u = 1$
$x = \frac{\pi}{4},\ u = \frac{1}{\cos\frac{\pi}{4}} = \frac{2}{\sqrt{2}}$

$\int_0^{\pi/4} \tan x \sec^2 x\, dx = \int_1^{\frac{2}{\sqrt{2}}} u\, du = \frac{1}{2}u^2\Big]_1^{2/\sqrt{2}} = \frac{1}{2}\left(\frac{4}{2} - 1\right)$
$= \frac{1}{2}$ //

continues . . .

(c) $\int_0^{\pi/2} sinx \cdot cosx \cdot e^{sinx}\, dx$

Let $u = \sin x \Rightarrow du = \cos x\, dx$.
when $x = 0$, $u = 0$
when $x = \frac{\pi}{2}$, $u = 1$

$= \int_0^1 \underbrace{u}_{g(u)} \underbrace{e^u}_{f'(u)}\, du$

$\Rightarrow g'(u) = 1 \quad f(u) = e^u$

$= ue^u]_0^1 - \int_0^1 e^u \cdot 1\, du$

$= 1e^1 - 0 \cdot e^0 - e^u]_0^1$

$= e^1 - (e^1 - e^0)$

$= 1$

(d) $\int \frac{x^2+4}{x^2-4}\, dx$

$$\begin{array}{r|l} & 1 \\ x^2-4 & x^2 \quad +4 \\ & x^2 \quad -4 \\ \hline & \quad\quad 8 \end{array}$$

$= \int 1 + \frac{8}{x^2-4}\, dx$

$= x + \int \frac{8}{(x-2)(x+2)}\, dx$

$= x + \int \frac{2}{x-2}\, dx + \int \frac{-2}{x+2}\, dx$

$= x + 2\ln|x-2| - 2\ln|x+2| + C$

write $\frac{8}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$

$= \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$

$\Rightarrow 8 = A(x+2) + B(x-2)$

$\xRightarrow{x=2} 8 = 4A \quad A = 2$

$\xRightarrow{x=-2} 8 = -4B \quad B = -2$

continues . . .

(e) $\int \frac{1}{(x+1)x^2}\, dx$

$= \int \frac{1}{x+1} dx - \int \frac{1}{x} dx + \int \frac{1}{x^2} dx$

$= \ln|x+1| - \ln|x| - \frac{1}{x} + C$

write $\frac{1}{(x+1)x^2} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x^2}$

$= \frac{Ax^2 + Bx(x+1) + C(x+1)}{(x+1)x^2}$

$1 = Ax^2 + Bx(x+1) + C(x+1)$

$\xRightarrow{x=0} 1 = C$

$\xRightarrow{x=-1} 1 = A$

$\xRightarrow{x=1} 1 = 1 + B \cdot 1 \cdot 2 + 1 \cdot 2 \Rightarrow B = -1$

13. (6 marks) Determine whether the following integral is convergent. If the integral is convergent, compute its value.

$$\int_0^\infty \frac{1}{\sqrt{x+1}}\, dx$$

$= \lim_{z \to \infty} \int_0^z (x+1)^{-\frac{1}{2}}\, dx$

$= \lim_{z \to \infty} \frac{1}{\frac{1}{2}} (x+1)^{\frac{1}{2}}]_0^z$

$= \lim_{z \to \infty} 2((z+1)^{\frac{1}{2}} - 1)$

$= \infty$

$\therefore$ the integral is divergent.

continues . . .

14. (6 marks) Solve the following system of linear equations.

$$\begin{aligned} 5x &- y + 2z = 6 \\ x &+ 2y - z = -1 \\ 3x &+ 2y - 2z = 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 6 \\ 1 & 2 & -1 & -1 \\ 3 & 2 & -2 & 1 \end{array}\right] \xRightarrow{row3-3\times row2} \left[\begin{array}{ccc|c} 5 & -1 & 2 & 6 \\ 1 & 2 & -1 & -1 \\ 0 & -4 & 1 & 4 \end{array}\right]$$

$$\xRightarrow{5\times row2 - row1} \left[\begin{array}{ccc|c} 5 & -1 & 2 & 6 \\ 0 & 11 & -7 & -11 \\ 0 & -4 & 1 & 4 \end{array}\right]$$

$$\xRightarrow{11row3+4row2} \left[\begin{array}{ccc|c} 5 & -1 & 2 & 6 \\ 0 & 11 & -7 & -11 \\ 0 & 0 & -17 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} 5x - y + 2z = 6 \\ 11y - 7z = -11. \\ -17z = 0. \end{cases}$$

$$\Rightarrow z = 0,$$

$11y = -11 + 7\cdot 0.$ and $y = -1$

$5x - (-1) + 2\cdot(0) = 6.$

$5x = 5$

$x = 1$

Ans $x = 1,\ y = -1,\ z = 0$

continues . . .

15. (12 marks) Suppose A and B are events with P(A)=0.75, P(B)=0.4 and P(A ∪ B)= 0.9, find

(i) $P(A^c)$ $= 1 - P(A) = 0.25$

(ii) P(A ∩ B)

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\Rightarrow P(A\cap B) = P(A) + P(B) - P(A\cup B) = 0.75 + 0.4 - 0.9 = 0.25$

(iii) P(A|B) $= \frac{P(A\cap B)}{P(B)} = \frac{0.25}{0.4} = \frac{5}{8} = 0.625$

(iv) P(B|A) $= \frac{P(A\cap B)}{P(A)} = \frac{0.25}{0.75} = \frac{1}{3}.$

(v) Are A and B independent? Explain.

no because $P(A\cap B) \neq P(A)\,P(B)$

(vi) Are A and B mutually exclusive? Explain.

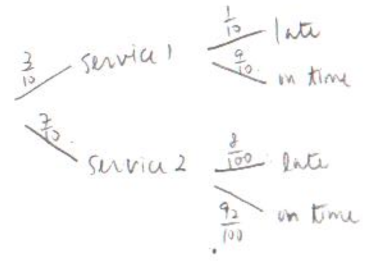
no because $P(A\cap B) > 0$

16. (8 points) Assume that 30% of all plants in a field are infested with aphids. Suppose that you pick 10 plants at random. What is the probability that none of them carried aphids?

Ans $= \left(\frac{7}{10}\right)^{10}.$

continues . . .

17. (8 marks) Two shipping services offer overnight delivery of parcels, and both promise delivery before 10 A.M. A mail order catalog company ships 30% of its overnight packages using service 1 and 70% using service 2. Service 1 fails to meet the 10 A.M. delivery promise 10% of the time, whereas Service 2 fails to deliver by 10 A.M. 8% of the time. Suppose that you made a purchase from this company and were expecting your package by 10 A.M., but it is late. Find the probability that the parcel was mailed by service 1.



$$P(\text{late}) = \frac{3}{10}\cdot\frac{1}{10} + \frac{7}{10}\cdot\frac{8}{100}$$
$$= \frac{86}{1000}$$

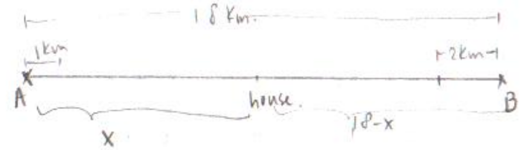
$$P(\text{service 1 and late}) = \frac{3}{10}\cdot\frac{1}{10}$$
$$= \frac{3}{100}$$

$$P(\text{service 1} \mid \text{late}) = \frac{\frac{3}{100}}{\frac{86}{1000}} = \frac{30}{86} = \frac{15}{43}$$

continues . . .

18. (8 marks) Two industrial plants A and B are located 18km apart, and emit 80 ppm (part per million) and 720 ppm of particulate matter, respectively. Plant A is surrounded by a restricted area of 1 km, in which no housing is allowed, while the restricted area around plant B has a radius of 2 km. The concentration of particulate matter arriving at any other point Q from each plant decreases proportional to the reciprocal of the distance between that plant and Q. Where should a house be located on a road joining the two plants to minimize the total concentration of particulate matter arriving from both plants?
[Recall that the reciprocal of a number a is $\frac{1}{a}$]

max/min.



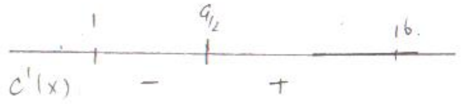
$1 < x < 16$

Concentration
$$C(x) = 80\cdot\frac{1}{x} + 720\cdot\frac{1}{18-x}$$

Minimize C(x)

$$C'(x) = -\frac{80}{x^2} + \frac{720}{(18-x)^2}$$
$$= \frac{-80(18-x)^2 + 720x^2}{x^2(18-x)^2}$$
$$= -80\frac{((18-x)^2 - 9x^2)}{x^2(18-x)^2}$$
$$= \frac{-80\cdot(18-4x)(18+2x)}{x^2(18-x)^2}$$

$C'(x) = 0 \Rightarrow x = \frac{9}{2}$ or $x = -9 \rightarrow$ but $x > 1$



The End

C(x) decreasing increasing

∴ C(x) is minimized at $x = \frac{9}{2}$

The house should be located at $\frac{9}{2}$ km from plant A to minimize the total concentration.