Test 4

First Name:________________ Last Name:________________

Student No.:________________

There are 4 long answer questions (Q2-Q5). **You must show all your work.**

(non-programmable, non-graphing calculators)

Q1 (8 Marks) If $\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$, then

$\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = \underline{35}$

$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \underline{7}$

$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = \underline{-7}$

$\det(2A) = \underline{2^3 \times 7} = 56$

Q2 (6 Marks)Use the cramer's rule to compute the solutions of the systems

$$3x_1 - 2x_2 = 7$$

$$-5x_1 + 6x_2 = -5$$

sol: $A = \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$ ← $\boxed{2 \text{ marks}}$

$$x_1 = \frac{\det\begin{bmatrix} 7 & -2 \\ -5 & 6 \end{bmatrix}}{\det\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}} = \frac{42-10}{18-10} = 4 \leftarrow \boxed{2 \text{ marks}}$$

$$x_2 = \frac{\det\begin{bmatrix} 3 & 7 \\ -5 & -5 \end{bmatrix}}{\det\begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}} = \frac{-15-(-35)}{18-10} = \frac{5}{2} \leftarrow$$

$\boxed{2 \text{ marks}}$

Q3 (6 Marks)Let $z_1 = 2 + 5i$, $z_2 = 3 - 2i$. Find $\operatorname{Re} z_1$, $\operatorname{Im} z_2$. and $\frac{z_1}{z_2}$.

Sol: $Re\, z_1 = 2$ (1) $Im\, z_2 = -2$ (1)

$$\frac{z_1}{z_2} = \frac{2+5i}{3-2i} = \frac{(2+5i)(3+2i)}{(3-2i)(3+2i)} \quad (2)$$

$$= \frac{6+4i+15i+10i^2}{9-4i^2} \quad (1)$$

$$= \frac{-4+19i}{13} = \frac{-4}{13} + \frac{19}{13}i \quad (1)$$

Q4 (6 Marks) Find the eigenvalues of matrix A= $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

Sol: $\det(A - \lambda I) = 0$ (1)

$$\left.\begin{aligned} \begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} &= (1-\lambda)(3-\lambda) - 1\times(-2) \\ &= 3 - 4\lambda + \lambda^2 + 2 = \lambda^2 - 4\lambda + 5 = 0 \end{aligned}\right\} \quad (2)$$

$$\lambda = \frac{4 \pm \sqrt{4^2 - 4\times 5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2\sqrt{-1}}{2} \quad (1)$$

$\Rightarrow \lambda_1 = 2 + i$ (1) $\qquad \lambda_2 = 2 - i$ (1)

Q5 Given matrix A=$\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$?

(a) (6 Marks) Find all eigenvalues and eigenvectors.

(b)(2 Marks) Is A diagonalizable? Why?

(c) (6 Marks) Compute A^3.

Sol: (a) $\det(A-\lambda I)=0$

$$\left.\begin{aligned} \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} &= (4-\lambda)(-1-\lambda)-2\times(-3) = \lambda^2-3\lambda+2 \\ &= (\lambda-2)(\lambda-1) \end{aligned}\right\} \boxed{3 \text{ marks}}$$

$\lambda_1 = 2 \qquad \lambda_2 = 1$

$$\left.\begin{aligned} &\text{For } \lambda_1=2. \quad \text{solve } (A-2I)\vec{x}=\vec{0} \Rightarrow \vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ &\text{For } \lambda_2=1 \quad \text{solve } (A-2I)\vec{x}=\vec{0} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}\right\} \text{3 mark}$$

(b) Yes. (1 mark)

since A has 2 independent eigenvectors. (1 mark).

(c) From (a) $A = PDP^{-1}$

where $P=\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (1 mark) $\left(\text{or } P=\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\right)$

$D=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (1 mark) $\left(D=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)$

$P^{-1}=\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ (1 mark) $\left(P^{-1}=\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}\right)$

$$A^3 = PD^3P^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2^3 & 0 \\ 0 & 1^3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \quad \text{(2 marks)}$$

$$= \begin{bmatrix} 22 & -21 \\ 14 & -13 \end{bmatrix} \quad \text{(1 mark).}$$