Q1.7:

(a) $\left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1/2 & 2 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1 & -3/2 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & -1 & -3 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 5 & 3 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$

Therefore, $A^{-1} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}$

(b) $E_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_4 = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$E_7 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$,

we know that $A = E_1^{-1} E_2^{-1} \cdots E_8^{-1}$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Q1.9: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/5 \end{bmatrix}$

In general, the diagonal matrix $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$ is invertible if and only if $d_1 \neq 0, d_2 \neq 0, \cdots d_n \neq 0$ and in this case the inverse is $D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$

Q 1.12. using theorem 1.12, this is the case if and only if A is invertible, if and only if A is row equivalent to the identity matrix:

$$A=\begin{pmatrix} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ -1 & 2 & 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -5 & -4 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 5 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -5 & -4 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we see that A is <u>not</u> invertible, so the span of its column vectors is <u>not</u> all of $\mathbb{R}^4$.

Q 1.14: $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}$$

Q 1.16: $$AC = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{pmatrix} \Longleftrightarrow \underbrace{(A^{-1}A)}_{I}C = A^{-1}\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{pmatrix}$$

$$\Longleftrightarrow C = A^{-1}\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 4 & 4 \\ 12 & 11 \end{pmatrix}$$

Q 1.21: Since $$\begin{pmatrix} 2 & 4 & 2 \\ 1 & r & 3 \\ 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & r-2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & r \end{pmatrix},$$

the matrix is invertible for any value of r except $r=0$ (if $r=0$, then A is not row equivalent to the identity matrix)

(left to right was proved in assignment 4)

Q 1.22: If $B = CA$ and C is invertible, then C may be expressed as a product of elementary matrices as shown in the hint provided, that is,

$$B = CA = E_1^{-1} E_2^{-1} \cdots E_t^{-1} A$$

and $E_1^{-1}, E_2^{-1}, \cdots, E_t^{-1}$ are all elementary matrices.
(recall that the inverse of an elementary matrix exists and is an elementary matrix itself)

Hence, B and A are row equivalent.

Q 1.27: (a) note that $A(A^{-1}B) = (AA^{-1})B = IB = B$. so, $X = A^{-1}B$ is a solution. To show uniqueness, suppose $AX = B$. Then $A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B$

so this is the only solution.

(b) let $E_1, E_2, \cdots, E_t$ be the elementary matrices that reduce $[A|B]$ to $[I|X]$, and let $C = E_t \cdots E_2 E_1$.
Then $CA = I$ and $CB = X$. thus $C = A^{-1}$ and $X = A^{-1}B$.

Q 2.6: let $P = \{ (x, y, z) \mid x, y, z \in \mathbb{R}, x = 2y + z \}$ which is a nonempty subset of $\mathbb{R}^3$. let $\vec{v} = [2a+b, a, b]$ and $\vec{w} = [2c+d, c, d]$ be in P. Then $\vec{v} + \vec{w} = [2a+b+2c+d, a+c, b+d]$
$= [2(a+c) + (b+d), a+c, b+d]$ which has the form $[2y+z, y, z]$ and is in P. (closed under vector addition)

Also, $r[2a+b, a, b] = [2ra+rb, ra, rb]$ which is in P
(closed under scalar multiplication)

Thus P is a subspace of $\mathbb{R}^3$.

<u>Q 2.8</u>: Let $P = \{[2x, x+y, y] \mid x, y \in \mathbb{R}\}$ which is a <u>nonempty</u> subset of $\mathbb{R}^3$. Let $\vec{v} = (2a, a+b, a)$ and $\vec{w} = [2c, c+d, d]$ be in P. Then

$$\vec{v}+\vec{w} = [2a+2c,\ a+b+c+d,\ b+d]$$
$$= [2(a+c),\ (a+c)+(b+d),\ b+d]$$

which has the form $[2x, x+y, y]$ and is in P.

Also, $r[2a, a+b, b] = [2ra, ra+rb, rb]$ which is in P. Thus P is a subspace of $\mathbb{R}^3$.

<u>Q 2.13</u>: (a) Every subspace of $\mathbb{R}^2$ is either $\{\vec{0}\}$, all vectors along a line through the origin, or all of $\mathbb{R}^2$.

(b) Every subspace of $\mathbb{R}^3$ is either $\{\vec{0}\}$, all vectors along a line through the origin, all vectors in a plane through the origin, or all of $\mathbb{R}^3$.

<u>Q 2.14</u>: Let W be a subspace of $\mathbb{R}^n$. Then W is nonempty, so we can choose $\vec{w} \in W$. Then $0\vec{w} \in W$ by the closure property for scalar multiplication. Since $0\vec{w} = \vec{0}$, this shows that $\vec{0} \in W$.

Q2.15: No. $\{\vec{0}\}$ is NOT a basis for the subspace $\{\vec{0}\}$ of $\mathbb{R}^n$, because $2\cdot\vec{0} = 3\cdot\vec{0} = \vec{0}$, so $\vec{0}$ is NOT a unique linear combination of $\vec{0}$, even though

$$\{\vec{0}\} = sp(\vec{0})$$

Q2.17: Recall the procedure we discussed in the class:

$$\begin{bmatrix} 3 & 1 & 1 \\ 6 & 2 & 2 \\ -9 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} -r/3 - s/3 \\ r \\ s \end{bmatrix}$$

$$= r\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the solution space. (Note that the basis is not unique. $\vec{x}$ can be written as $\vec{x} = \frac{r}{3}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \frac{s}{3}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$

$$= r_1\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + r_2\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

where $r_1 = r/3$, $r_2 = s/3$.
so $\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$ is also a basis for solution space)

Q2.20: $\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 1 & 3 & 0 \\ 1 & -1 & -7 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -7 & 2 \\ 0 & 4 & 20 & -4 \\ 0 & 2 & 10 & -2 \\ 0 & 3 & 15 & -3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 2r - s \\ -5r + s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

so $\{ (2,-5,1,0), (-1,1,0,1) \}$ is a basis for the solution space.

Q 2.22: We can use Theorem 1.15 or take advantage of the equivalence between parts (2) and (3) of Theorem 1.17.

Let $\vec{v} = [-1,1)$ and $\vec{w} = [1,2]$. Using Theorem 1.15, we need only test whether $r\vec{v} + s\vec{w} = \vec{0}$ implies $r = s = 0$. We solve this homogeneous system of equations (see hint) by the reduction:

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right] \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \end{array}\right)$$ the matrix is therefore

invertible, so $r = s = 0$ is the unique solution. so

$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $sp\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$.

(and of course a basis for $\mathbb{R}^2$ by Theorem 1.16).

Q 2.25: Similarly

$$\left[\begin{array}{ccc|c} 2 & 4 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ -3 & 2 & 3 & 0 \end{array}\right] \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

The matrix is invertible, so the unique solution is $r_1 = r_2 = r_3 = 0$, therefore by Theorem (1.15),

$\left\{ \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$ is a basis for $sp\left(\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right)$.

Q 2.32: Use the procedure discussed in the class:

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -6 & -6 & -12 \\ 0 & -7 & -7 & -14 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ The solution space is given

by $\begin{bmatrix} -2s-r \\ -s-2r \\ s \\ r \end{bmatrix}$ for all scalars r and s. Thus a basis is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Q 2.42: By Theorem 1.13, If a homogeneous system has a nontrivial solution $\vec{h}$, then $r\vec{h}$ is also a solution for each $r \in \mathbb{R}$. Thus the solution set is either $\{\vec{0}\}$ or infinite in case there exists a nontrivial solution $\vec{h}$.

Q 2.45(a): Recall: The set A is a subset of the set B, denoted by $A \subseteq B$, if every element of A is also in B.

Now two sets are equal, that is $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Now we want to prove that $sp(\vec{v}_1, \vec{v}_2) = sp(\vec{v}_1, 2\vec{v}_1 + \vec{v}_2)$

- First prove that $sp(\vec{v}_1, 2\vec{v}_1 + \vec{v}_2) \subseteq sp(\vec{v}_1, \vec{v}_2)$:

  Let $\vec{w}$ be any element in $sp(\vec{v}_1, 2\vec{v}_1 + \vec{v}_2)$, then

  $\vec{w} = a\vec{v}_1 + b(2\vec{v}_1 + \vec{v}_2)$ for some scalars a and b.

  We then have $\vec{w} = (a + 2b)\vec{v}_1 + b\vec{v}_2$

  which shows that $\vec{w} \in sp(\vec{v}_1, \vec{v}_2)$

  $\therefore sp(\vec{v}_1, 2\vec{v}_1 + \vec{v}_2) \subseteq sp(\vec{v}_1, \vec{v}_2)$ (1)

- Next prove that $sp(\vec{v}_1, \vec{v}_2) \subseteq sp(\vec{v}_1, 2\vec{v}_1 + \vec{v}_2)$

  Let $\vec{w} \in sp(\vec{v}_1, \vec{v}_2)$ then $\vec{w} = a\vec{v}_1 + b\vec{v}_2$ for some scalars a and b.

  So $\vec{w} = a\vec{v}_1 + b\vec{v}_2 = a\vec{v}_1 + b(\underbrace{-2\vec{v}_1 + 2\vec{v}_1}_{\vec{0}} + \vec{v}_2)$

  $= (a - 2b)\vec{v}_1 + b(2\vec{v}_1 + \vec{v}_2)$

  which shows that $\vec{w} \in sp(\vec{v}_1, 2\vec{v}_1 + \vec{v}_2)$

$\therefore$ $sp(\vec{v}_1, \vec{v}_2) \subseteq sp(\vec{v}_1, 2\vec{v}_1+\vec{v}_2)$ ②

①, ② $\Rightarrow$ $sp(\vec{v}_1, \vec{v}_2) = sp(\vec{v}_1, 2\vec{v}_1+\vec{v}_2)$

Q 2.46 (a):

In view of Theorem 1.15 and Q 2.45 (a), It remains to show that the zero vector is a unique l.c. of $\vec{v}_1$ and $2\vec{v}_1+\vec{v}_2$.

suppose $c_1\vec{v}_1 + c_2(2\vec{v}_1+\vec{v}_2) = \vec{0}$

Then $(c_1+2c_2)\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$. Because $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $sp(\vec{v}_1, \vec{v}_2)$, Theorem 1.15 implies that $c_1+2c_2=0$ and $c_2=0$. These show that $c_1=c_2=0$, which is what we need to show.

Q 2.47: clearly $W_1 \cap W_2$ is nonempty; it contains $\vec{0}$.

Let $\vec{v}, \vec{w} \in (W_1 \cap W_2)$, Then $\vec{v}, \vec{w} \in W_1$ and $\vec{v}, \vec{w} \in W_2$ so $\vec{v}+\vec{w} \in W_1$ and $\vec{v}+\vec{w} \in W_2$ since W_1 and W_2 are subspaces. Thus $\vec{v}+\vec{w} \in (W_1 \cap W_2)$. Similarly $r\vec{v} \in W_1$ and $r\vec{v} \in W_2$ since W_1 and W_2 are subspaces, so $r\vec{v} \in (W_1 \cap W_2)$. Thus $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$, because we proved that $W_1 \cap W_2$ is 1) nonempty 2) closed under vector addition and 3) closed under scalar multiplication.