1. (10 pts) For each of the following equations, state the order of the equation, whether it is nonlinear, linear inhomogeneous or linear homogeneous. You do *not* have to justify your answer:

$$u_{tt} + u_{xxxx} = 0, \tag{a}$$
$$u_t + uu_x + u_{xxx} = 0, \tag{b}$$
$$u_{ttxx} + u_{ttyy} + u_{ttzz} + u_{zz} = \cos(t). \tag{c}$$

2. (20 pts) Consider the first order equation:

$$u_t + tu_x = 0. \tag{1}$$

(i) Find the characteristic curves and sketch them in the (x,t) plane.

(ii) Write the general solution.

(iii) Solve equation (1) with initial condition $u(x,0) = \sin(x)$. Explain why the solution is fully determined by the initial condition.

3. (15 pts) Solve the wave equation with the following initial conditions

$$\begin{cases} u_{tt} - 16u_{xx} = 0, & -\infty < x < \infty \\ u(x,0) = \dfrac{1}{2+\cos(x)}, & \\ u_t(x,0) = \dfrac{\sin(x)}{2+\cos(x)} & \end{cases}$$

4. (20 pts) Consider the PDE with boundary conditions:

$$\begin{aligned} &u_{tt} - c^2u_{xx} + \omega^2 u = 0, \qquad 0 < x < L, \\ &(u_x - \alpha u_t)(0,t) = 0, \\ &(u_x + \beta u_t)(L,t) = 0, \end{aligned}$$

where $\alpha > 0, \beta > 0$ are constants. Prove that the energy $E(t)$ defined as

$$E(t) = \frac{1}{2}\int_0^L \left(u_t^2 + c^2u_x^2 + \omega^2u^2\right) dx$$

is non-increasing function of t.

Is it necessarily strictly decreasing?

5. (20 pts) Check that $u = xt + \frac{1}{12}x^3$ satisfies diffusion equation $u_t - 2u_{xx} = 0$ and find

$$\begin{aligned} M(T) &= \max_{0\le x\le L, 0\le t\le T} u(x,t), \\ m(T) &= \min_{0\le x\le L, 0\le t\le T} u(x,t). \end{aligned}$$

Where is the maximum value $u(x,t) = M(T)$ achieved? Where is the minimum $u(x,t) = m(T)$ achieved? Verify the maximum and minimum principle.

6. (20 pts) Write the solution of the diffusion equation on a half line

$$\begin{aligned} &u_t - ku_{xx} = 0, && 0 < x < +\infty, \\ &u(x,0) = e^{-2x}, \\ &u(0,t) = 0 \end{aligned}$$