University of Toronto

MAT237Y1Y TERM TEST 2

Thursday, Jan 12, 2011

Duration: 90 minutes

No aids allowed

Instructions: There are 12 pages including the cover page and the extra sheet at the back. Please answer all questions in the spaces provided (if you use back of a sheet or the extra sheet, please clearly specify that.) Document your arguments by briefly stating the results you use (in most cases you may find relevant definitions or results in an earlier question of this test.) The value for each question is indicated in parentheses beside the question number. The test is out of 100 but there is a total of 112 marks available of which 12 marks are bonus marks.

NAME: (last, first) ________________

STUDENT NUMBER: Marking scheme

SIGNATURE: ________________

CHECK YOUR TUTORIAL:

◯ TUT01 M3-4		◯ TUT03 T2-3
◯ TUT04 W3-4	◯ TUT05 W5-6	◯ TUT06 R5-6

#5C page 11

$S = \{(x,y) : x^2+y^2=1\}$

$\nexists$ any cont. function f ...

$f : S \to \mathbb{R}$

MARKER'S REPORT:

Question	MARK
Q1	/16
Q2	/30
Q3	/19
Q4	/22
Q5	/25
TOTAL	/112

1. Critical points

a) (4 marks) What does it mean to be a critical point for a differentiable function $f(x, y, z)$? Determine the critical point(s) of the function $f(x, y, z) = x^2 - 3(y-2)^2 - 5(z-1)^2$

$\mathbb{a}$ is a critical pt of f if $\nabla f(\mathbb{a}) = \mathbb{0}$. (1)

$$\nabla f(\mathbb{x}) = \begin{bmatrix} 2x \\ -6(y-2) \\ -10(z-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x = 0 \\ y = 2 \\ z = 1 \end{matrix}$$ (2)

$\mathbb{a} = (0, 2, 1)$ is a critical pt of f. (1)

b) (6 marks) For each critical point in part (a), give Taylor's polynomial of degree 2 for the function f at the critical point (in both matrix and expanded forms.)

$$P_{\mathbb{a},2}(\mathbb{h}) = f(\mathbb{a}) + \nabla f(\mathbb{a}) \cdot \mathbb{h} + \frac{1}{2}\mathbb{h}^T H(\mathbb{a})\mathbb{h}$$ (2)

$\mathbb{h} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$, $f(\mathbb{a}) = 0$, $\nabla f(\mathbb{a}) = 0$, $H(\mathbb{a}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -10 \end{bmatrix}$

$$= \frac{1}{2}[h_1\ h_2\ h_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$ (2)

$$= \frac{1}{2}(2h_1^2 - 6h_2^2 - 10h_3^2) = h_1^2 - 3h_2^2 - 5h_3^2$$ (2)

c) (6 marks) Classify the nature of each critical point obtained in part (a). (Justify your answer.)

Since $f(a+h) - f(a) = \frac{1}{2} h^T H(a) h + R(h)$ ①

and R can be made smaller than … as $h \to 0$

The decision about sign of … is upon $\frac{1}{2} h^T H(a) h = h_1^2 - 3h_2^2 - 5h_3^2$

We select two paths: $h = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $k = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ②

and observe that $\frac{1}{2} h^T H(a) h = 1 > 0$ ①, but $\frac{1}{2} k^T H(a) k = -3 < 0$ ①

so the value of f increase along direction $h = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and decreases along $k = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. This means that a is a saddle pt. ①

<u>alternate sln</u>

Eigen values of H are $2, -6, -10$ ① , why? ②

and since some are >0 and some are <0 then a is a saddle pt ② by Theorem ? ①

③

2. Differentiability and linear approximation

a) (17 marks) State the definition of differentiability for a function $f(x, y)$ at a point (a, b) in its domain. Use this definition to demonstrate that the function $f(x, y) = x|y|$ is differentiable at any point (a, b) with $b > 0$ and that it is not differentiable whenever $b = 0, a > 0$. Is the function differentiable at $(0, 0)$? Justify your answer.

(3)

– f is diff at $\mathbf{a} = (a, b)$ if $\exists\, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ s.t. $\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c} \cdot \mathbf{h}}{\|\mathbf{h}\|} = 0$

– if $b > 0$ $\quad f\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\right) - f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) - [c_1\ c_2]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = (a+h_1)(b+h_2) - ab - c_1h_1 - c_2h_2$ (1)

as $h_2 \to 0$ $\quad |h_2| < b$ so $b + h_2 > 0$

$= ah_2 + bh_1 - c_1h_1 - c_2h_2 + h_1h_2 = (a - c_2)h_2 + (b - c_1)h_1 + h_1h_2$

so if we let (2) $a = c_2$ and $b = c_1$ Then $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c}\cdot\mathbf{h} = h_1h_2$ (1)

and $\lim \frac{h_1h_2}{\|\mathbf{h}\|} = 0$ (1) either by polar or $-1 \le \frac{h_1}{\|\mathbf{h}\|} \le 1$ (2) so $\frac{h_1h_2}{\|\mathbf{h}\|} \to 0$ as $\mathbf{h} \to \mathbf{0}$

– at $(0,0)$: $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - [c_1\ c_2]\mathbf{h} = h_1|h_2| - c_1h_1 - c_2h_2$ let $c_1 = 0 = c_2$, Then

$\exists\, \mathbf{c}$ That satisfies def of diff b/c $\lim \frac{h_1|h_2|}{\|\mathbf{h}\|} = 0$ (2)

– if $b = 0$ and if $\mathbf{c} = [c_1\ c_2]$ Then $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c}\mathbf{h} = a|h_2| + h_1|h_2| - c_1h_1 - c_2h_2$

if f is to be diff at $(a, 0)$ Then $\lim_{\mathbf{h} \to \mathbf{0}} \frac{}{\|\mathbf{h}\|} = 0$ so along $\mathbf{h} = \begin{bmatrix} h_1 \\ 0 \end{bmatrix}$ This

limit becomes $-\frac{c_1h_1}{\|\mathbf{h}\|} = 0$ and This implies $c_1 = 0$. (2) so we are left with

$\|\mathbf{h}\| \to |h_1|$ & $\frac{h_1}{|h_1|} \to +1, -1$

$\lim \frac{a|h_2| - c_2h_2 + h_1|h_2|}{\|\mathbf{h}\|}$ now along $\mathbf{h} = \begin{bmatrix} 0 \\ h_2 \end{bmatrix}$ (3) we have $\lim_{h_2 \to 0} \frac{a|h_2| - c_2h_2}{|h_2|} = 0$

$= \lim_{h_2 \to 0} \left[a - \frac{c_2h_2}{|h_2|}\right] = 0$ only when $c_2 = 0$ & $a = 0$, not if $a \neq 0$. so for $a > 0$ not diff.

b) (6 marks) With the function $f(x, y) = x|y|$, as in part (a), determine the maximum value of $\partial_{\boldsymbol{u}} f(1, -1)$, for all unit vectors $\boldsymbol{u}$.

f is diff for $b \neq 0$ and near $b = -1$ (1) $f(x,y) = -xy$ and

$\nabla f(1,-1) = \begin{bmatrix} -y \\ -x \end{bmatrix}\Big|_{(1,-1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. (1) We know $\partial_u f(x) = \nabla f(x) \cdot u$, so (1)

and $|\nabla f(a) \cdot u|$ is maximized when $\nabla f(a) \parallel u$ (1) so for $u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ we have

$|\nabla f(1,-1) \cdot u| = \frac{1}{\sqrt{2}} [1 \ -1] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{2}{\sqrt{2}} = \sqrt{2}$ ← max value for $\partial_u f(1,-1)$

(2)

c) (7 marks) For the function f as in (a) and (b), use the differential $df(\boldsymbol{a}; \boldsymbol{h})$ with $\boldsymbol{a} = (3, 4)$ to approximate the value of $f(2.9, 4.2)$.

$a =$ near $(3,4)$ $f(x,y) = xy$

$df(a; h) = \nabla f(a) \cdot h = [4 \ 3] \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix} = -0.4 + 0.6 = 0.2$

(2) (1)

& $\nabla f(3,4) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (1)

let $h = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}$ (1)

now $f(2.9 + 4.2) = f(a+h) \overset{(1)}{\approx} f(a) + df(a;h)$

(1) $= 12 + 0.2 = 12.2$

~~TBK~~

3. Lagrange's method

a) (4 marks) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\boldsymbol{g} : \mathbb{R} \longrightarrow \mathbb{R}^n$ be differentiable. Use Chain rule to express $\frac{d}{dt} f(\boldsymbol{g}(t))$.

$\frac{d}{dt} f(g(t)) = \nabla f(g(t)) \cdot g'(t)$ (4)

b) (6 marks) Let $G : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at a point $\boldsymbol{a}$ with $\nabla G \neq 0$ at $\boldsymbol{a}$. Suppose that f, as a function on the set $S = \{\boldsymbol{x} : G(\boldsymbol{x}) = 0\}$ has a local extreme at $\boldsymbol{x} = \boldsymbol{a}$. Prove that $\nabla f(\boldsymbol{a}) = \lambda \nabla G(\boldsymbol{a})$ for some real λ. (You may assume that for any direction vector $\boldsymbol{u}$ perpendicular to ∇G, there is a curve $\boldsymbol{g} : [-1, 1] \longrightarrow S \subset \mathbb{R}^n$ with $\boldsymbol{g}(0) = \boldsymbol{a}$, $\boldsymbol{g}'(0) = \boldsymbol{u}$.)

let $g(t)$ be a diff curve & $g(0) = a$ be in S, so $\varphi(t) = f(g(t))$ has a local (1) max at 0, That is $\varphi'(0) = 0$, or (1) $\nabla f(g(0)) \cdot g'(0) = 0$

$\nabla f(a) \cdot g'(0) = 0$

so $\nabla f(a)$ is perpendicular to any $g'(0)$, (1) → tangent vector to S

and we already know $\nabla G(a)$ is normal to S at a, (1) Then

$\nabla f(a) \parallel \nabla G(a)$ or $\exists \lambda$ st. $\nabla f(a) = \lambda \nabla G(a)$

(1) (1)

(2) extra mark for proving $\nabla G(a)$ is perp to S.

c) (9 mark) Use the result of part (b) to determine all the points of the set $S = \{(x,y) : x^2 + y^2 = 1\}$, at which the function $z = f(x,y) = y^2$ has maximum or minimum values.

$\nabla f(x) = \begin{bmatrix} 0 \\ 2y \end{bmatrix}$ (1) $\quad G(x,y) = x^2 + y^2 - 1 \quad \nabla G(x) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ (1)

Solve $\begin{cases} \nabla f(x) = \lambda \nabla G(x) \\ G(x) = 0 \end{cases}$ (1) $\quad \begin{cases} 0 = \lambda 2x \\ 2y = \lambda 2y \\ x^2 + y^2 - 1 = 0 \end{cases}$ (1)

if $y \neq 0$ Then $\lambda = 1$ & $x = 0$ so $y = \pm 1$ (1.5) $(0,1), (0,-1)$ → $y^2 = 1$ max

if $y = 0$ Then $\lambda = 0$ & $x = \pm 1$ so $(1,0), (-1,0)$ (1.5) → $y^2 = 0$ min

(2)

4. The Mean Value Theorem

a) (4 marks) State the Mean Value Theorem for a differentiable function $f : S \longrightarrow \mathbb{R}$, where S is a convex subset of $\mathbb{R}^n$.

note: all the conditions of MVT are present in, so we only need to present the conclusion

for any $a, b \in S$ (1) $\exists\, c \in S$ (located on the line segment L connecting a to b) (1)

such that $f(b) - f(a) = \nabla f(c) \cdot (b-a)$ (2)

b) (10 marks) Prove the MVT as stated in part (a), using the MVT for the case $g : \mathbb{R} \longrightarrow \mathbb{R}$.

Let $h = b - a$ and $\varphi(t) = f(a + th)$, $0 \le t \le 1$ (3), so φ is diff and $\varphi(0) = f(a)$, $\varphi(1) = f(a+h) = f(b)$ (2)

now by MVT I $\exists\, u \in [0,1]$: $\dfrac{\varphi(1) - \varphi(0)}{1-0} = \varphi'(u) = \dfrac{d}{dt}\varphi(t)\Big|_{t=u} = \nabla f(a + uh) \cdot h$ (3)

$= \nabla f(c) \cdot (b - a)$ (1)

note $c = a + u(b-a)$, $0 \le u \le 1$, so c is on the line segment connecting a and b.

(1)

1~ ~2

c) (8 marks) Let $f(x,y) = x^2 + y^2$, $\boldsymbol{a} = (0,0)$, and $\boldsymbol{b} = (1,2)$. Determine a point $\boldsymbol{c}$ that satisfies the conclusion of the MVT for the function f and the points $\boldsymbol{a}$ and $\boldsymbol{b}$.

$f(\mathbb{b}) - f(\mathbb{a}) = 5 - 0$, (1) $\nabla f(\mathbb{x}) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$, (1) and $\mathbb{b} - \mathbb{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (1)

and L is $\mathbb{a} + t(\mathbb{b} - \mathbb{a})$

$= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1-0 \\ 2-0 \end{bmatrix} \right\}$

$= t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (1)

so we need $\mathbb{c}$ on L, That is

need $\mathbb{c} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (1) for some $0 \leq t \leq 1$

That satisfies $5 = \nabla f(\mathbb{c}) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

or $5 = [2t, 4t] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 10t$ so let $t = \frac{1}{2}$, $\mathbb{c} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

(3)

5. The Intermediate Value Theorem

a) (4 marks) State the Intermediate Value Theorem for a continuous function $f : S \longrightarrow \mathbb{R}$ where $S \subset \mathbb{R}^n$ is a connected set.

given (1) any $a, b \in S$ and any t that satisfies $f(a) < t < f(b)$ (1) or $f(b) < t < f(a)$

$\exists\, c \in S$ s.t. $f(c) = t$.

(1) (1)

b) (9 marks) Prove the IVT. In your proof you may use other theorems and definitions in the textbook; in this case state clearly the definitions and the theorems used in your proof.

Since cont. image of a connected set is connected (1), $f(S)$ (1) is a connected subset of $\mathbb{R}$. Any connected subset (1) of $\mathbb{R}$ is an interval, so $f(S)$ is an interval (1) $= I$. if $a, b \in S$

Then $f(a), f(b) \in I$ and $f(a) \le t < f(b) \Rightarrow t \in I$ (2)

definition of an interval.

so $t \in f(S)$. (1) Then (3) $\exists\, c \in S$ s.t. $t = f(c)$ (2)

definition $t \in f(S)$. (2)

$f : \mathbb{R}^2 \to \mathbb{R}$
or $S \to \mathbb{R}$

$S =$

c) (12 marks) Explain why the unit circle $\{(x, y) : x^2 + y^2 = 1\}$ is a connected set. Show that for any continuous function defined on the unit circle there must be a point (x_0, y_0) on the unit circle that satisfies
$f(x_0, y_0) = f(-x_0, -y_0)$.
(Hint: apply IVT to the function $g(x, y) = f(x, y) - f(-x, -y)$. Make sure to include all the details and justifications needed.)

unit Circle is pathwise Connected (2) b/c it is just a path $g(t) = (\cos t, \sin t)$

any pathwise Connected Set is Connected, so unit Circle is Connected. (1)

let $f : \mathbb{R}^2 \to \mathbb{R}$ ($S \to \mathbb{R}$) be Cont. and define $g(x,y) = f(x,y) - f(-x,-y)$

Choose any point on the unit Circle (1), say $(1,0)$. If $f(1,0) = f(-1,0)$ Then we are done. (1) Else, assume $f(1,0) - f(-1,0) > 0$ (1)

so $g(1,0) > 0$ clearly $g(-1,0) = f(-1,0) - f(-(-1),0) < 0$ (1)

so let $(1,0) = b$ and $(-1,0) = a$ (2) We have $g(a) < 0 < g(b)$

by IVT $\exists\, c \in S$ s.t. $g(c) = 0$ or $g(x_0, y_0) = 0$ (2)
($c \to (x_0, y_0)$)

or $f(x_0, y_0) - f(-x_0, -y_0) = 0$ or $f(x_0, y_0) = f(-x_0, -y_0)$

(1)