MATH 204: Vectors and Matrices - Dec. 2012

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Problem 1

Using the Gauss-Jordan method (i.e. reduced row echelon form method), find all the solutions of the following system of equations:

$$\begin{array}{rcl} 2x - 2y \phantom{{}-z} + 2u + 3v & = & 1 \\ 3x - 3y - z + 5u + 2v & = & 3 \\ 2x - 2y - 2z + 6u \phantom{{}+2v} & = & -2. \end{array}$$

Solution 1

Writing the system as an augmented matrix and performing elementary row operations,

$$\left(\begin{array}{ccccc|c} 2 & -2 & 0 & 2 & 3 & 1 \\ 3 & -3 & -1 & 5 & 2 & 3 \\ 2 & -2 & -2 & 6 & 0 & -2 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & -1 & 2 & -5/2 & 3/2 \\ 0 & 0 & -2 & 4 & -3 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 1 & -2 & 5/2 & -3/2 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{array}\right).$$

Therefore, letting $y = r$ and $u = s$ since they're free variables, we can write the solution to this system as:

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 5 + r - s \\ r \\ 6 + 2s \\ s \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 6 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r + \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} s.$$

Problem 2

Let $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$. a) Calculate M^{-1}; b) Find the matrix C such that $MC = B$.

Solution 2

a) To find the inverse of M, we can use the Matrix Inverse Algorithm:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right].$$

Therefore,

$$M^{-1} = \begin{bmatrix} 0 & 3 & -2 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

b)

$$MC = B \Rightarrow C = M^{-1}B = \begin{bmatrix} 0 & 3 & -2 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 2 & -1 \\ -1 & 4 \end{bmatrix}.$$

Problem 3

a) Use Cramer's Rule to solve the following system of equations:

$$\begin{array}{rcl} 2x + 3y & = & -2 \\ x + 3z & = & -1 \\ 2y + z & = & 2. \end{array}$$

b) Calculate the determinant of the matrix $\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 3 \\ 0 & 3 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$.

Solution 3

a) By Cramer's Rule, we have that

$$x = \frac{\begin{vmatrix} -2 & 3 & 0 \\ -1 & 0 & 3 \\ 2 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 2 & -2 & 0 \\ 1 & -1 & 3 \\ 0 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 2 & 3 & -2 \\ 1 & 0 & -1 \\ 0 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix}}.$$

We evaluate all four determinants by cofactor expansion, getting

$$\begin{vmatrix} -2 & 3 & 0 \\ -1 & 0 & 3 \\ 2 & 2 & 1 \end{vmatrix} = -3 \begin{vmatrix} -2 & 3 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} -2 & 3 \\ -1 & 0 \end{vmatrix} = -3(-4-6) + (0+3) = 33,$$

$$\begin{vmatrix} 2 & -2 & 0 \\ 1 & -1 & 3 \\ 0 & 2 & 1 \end{vmatrix} = -3 \begin{vmatrix} 2 & -2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = -3(4) + (-2+2) = -12,$$

$$\begin{vmatrix} 2 & 3 & -2 \\ 1 & 0 & -1 \\ 0 & 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} - \begin{vmatrix} 3 & -2 \\ 2 & 2 \end{vmatrix} = 2(0+2) - (6+4) = -6,$$

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix} = -3 \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3(4-0) + (0-3) = -15.$$

Therefore, the unique solution to our system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11/5 \\ 4/5 \\ 2/5 \end{pmatrix}.$$

b) Evaluating the determinant by cofactor expansion along the third column, we get

$$\begin{vmatrix} 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 3 \\ 0 & 3 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 2 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix}.$$

Evaluating each of these subdeterminants by cofactor expansion, we get

$$-2\left(\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}\right) + 3\left(\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix}\right) - \left(-2\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}\right).$$

This can easily be calculated, bringing us to

$$\begin{vmatrix} 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 3 \\ 0 & 3 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{vmatrix} = -2(2-12) + 3(2+4) - (6-2) = 34.$$

Problem 4

Let $\mathcal{L}$ be the line with parametric equations $x = 2+t, y = 1-t, z = 1+3t$, and let $\vec{v} = (1,2,0)$. Find vectors $\vec{w}_1, \vec{w}_2$ such that $\vec{v} = \vec{w}_1 + \vec{w}_2$, and such that $\vec{w}_1$ is parallel to $\mathcal{L}$ and $\vec{w}_2$ is perpendicular to $\mathcal{L}$.

Solution 4

Since vectors are independent of their initial position, we may imagine that $\vec{v}$ sits with its tail on $\mathcal{L}$. Thus, since we have the direction vector of $\mathcal{L}$ is $\vec{d} = (1,-1,3)$,

$$\vec{w}_1 = \text{proj}_{\mathcal{L}}\vec{v} = \frac{\vec{v}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d} = \frac{1}{11}(-1,1,-3)$$

and

$$\vec{w}_2 = \vec{v} - \vec{w}_1 = \frac{1}{11}(12,21,3).$$

Problem 5

Let $P_1 = (1,-1,1)$, $P_2 = (2,1,-1)$, and $P_3 = (1,-2,-1)$.

a) Find the area of the triangle with vertices P_1, P_2 and P_3.

b) Find an equation of the plane containing P_1, P_2 and P_3.

Solution 5

a) The formula for the area of a triangle is given by $A_\Delta = \frac{1}{2}||\vec{u}\times\vec{v}||$, where $\vec{u}$ and $\vec{v}$ are the vectors that determine the triangle. Fixing P_1, we see that the two vectors radiating from it (i.e. $\vec{P_1P_2}$ and $\vec{P_1P_3}$) are $\vec{u} = (1,2,-2)$ and $\vec{v} = (0,-1,-2)$. Thus,

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 0 & -1 & -2 \end{vmatrix} = \left(\begin{vmatrix} 2 & -2 \\ -1 & -2 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 0 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \right) = (-6,2,-1),$$

leading to

$$A_\Delta = \frac{1}{2}||\vec{u}\times\vec{v}|| = \frac{\sqrt{41}}{2}.$$

b) Since the equation of a plane is given by $\vec{n}\cdot(\vec{x}-\vec{x_0}) = 0$, where $\vec{x} = (x,y,z)$, we need to find a vector orthogonal to the plane and a point on the plane. We can take $\vec{n} = \vec{u}\times\vec{v}$ because it is perpendicular to both $\vec{u}$ and $\vec{v}$, which are vectors on the plane, and we can take $\vec{x_0}$ to be $\vec{P_1}$ since it's the point at the base of our chosen normal. Thus, we have

$$(-6,2,-1)\cdot(x-1,y+1,z-1) = 0 \Rightarrow -6x+2y-z = -9.$$

Problem 6

Let $\mathcal{L}$ be the line with parametric equations $x = 1 + 2t$, $y = 2 - 3t$, $z = 3 + t$, and let $\mathcal{P}$ be the plane $x + y + z - 10 = 0$.

a) Prove that $\mathcal{L}$ and $\mathcal{P}$ are parallel.

b) Find the distance between $\mathcal{L}$ and $\mathcal{P}$.

Solution 6

a) For a plane and a line to be parallel is the same as saying that the plane's normal vector, $\vec{n}$, and the line's direction vector, $\vec{d}$, are orthogonal, i.e. $\vec{n} \cdot \vec{d} = 0$. We find the plane's normal vector to be $\vec{n} = (1, 1, 1)$ and the line's direction vector to be $\vec{d} = (2, -3, 1)$ and, indeed, we have that $(1, 1, 1) \cdot (2, -3, 1) = 0$. Therefore, $\mathcal{L}$ and $\mathcal{P}$ are parallel.

b) This problem may be done via projections. Or, since the line and plane are parallel, the distance between them is the same as the distance between the plane and any point on the line. We already know the point $(1, 2, 3)$ from the equations of the line. So, we may apply this directly to the formula

$$d(P, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{4}{\sqrt{3}}.$$

Problem 7

Let $\vec{v}_1 = (2, -1, -2)$ and $\vec{v}_2 = (1, 2, -2)$.

a) Find scalars x and y such that $x\vec{v}_1 + y\vec{v}_2 = (5, -10, -2)$.

b) Find a vector $\vec{v}_3$ such that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$. Justify your answer.

Solution 7

a) To find these scalars we think of $x\vec{v}_1 + y\vec{v}_2 = (5, -10, -2)$ as being a system of equations with the unknowns being the scalars x and y:

$$\left(\begin{array}{cc|c} 2 & 1 & 5 \\ -1 & 2 & -10 \\ -2 & -2 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, the unique solution to the system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

b) To find the vector $\vec{v}_3$ such that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$, we adjoin our vectors $\vec{v}_1$ and $\vec{v}_2$ to the standard basis for $\mathbb{R}^3$ and use row operators to determine which of the standard basis vectors we may add to $\vec{v}_1$ and $\vec{v}_2$ while still maintaining linear independence:

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 5 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & -2 & -1 & -2 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{5}{6} \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{5}{6} \end{pmatrix}.$$

Thus, we will have linear independence if we choose $\vec{v}_3 = (1, 0, 0)$. Since we have found a set of three linear independent vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, it forms a basis for $\mathbb{R}^3$ (vector space with dimension 3).

Problem 8

Let $A = \begin{bmatrix} 1 & 0 & -2 & 0 & -1 & 3 \\ 0 & 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix}$. Find a basis for the solution space of the homogeneous system of linear equations $AX = 0$.

Solution 8

Finding a basis for the solution space is equivalent to finding the set of vectors whose span form the set of all possible solutions. Going from a matrix to equations, we see that we have

$$\begin{aligned} x - 2z - v + 3w &= 0 \\ y + 3z + 2v + w &= 0 \\ u + v - 2w &= 0. \end{aligned}$$

We notice that, since the leading ones in the matrix A are in the colums corresponding to variables x, y, u, that we have three free variables, a.k.a. three parametres. Call $z = r$, $v = s$, and $w = t$ We notice that we can write our solution, X, as the following:

$$X = \begin{bmatrix} 2r + s - 3t \\ -3r - 2s - t \\ r \\ -s + 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2r \\ -3r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ -2s \\ 0 \\ -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -3t \\ -t \\ 0 \\ 2t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} t,$$

therefore we have that

$$X = \text{span}\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We see automatically that a basis of the solution space of $AX = 0$ is given by

$$\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Linear independence isn't an issue but could be checked.

Problem 9

Find the standard matrices for the composition of two linear operators on $\mathbb{R}^2$: A rotation counterclockwise by 30° followed by a reflection about the y-axis.

Solution 9

All rotation matrices follow the same form, meaning that we simply need to plug in our value of θ:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

A reflection about the y-axis is a transformation that brings $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} -x \\ y \end{bmatrix}$. So we need to find the matrix that accomplishes this. It should be intuitive that the matrix is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If this isn't however, we may find it thus:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \Rightarrow \begin{cases} ax + by = -x \\ cx + dy = y \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 0 \\ c = 0 \\ d = 1 \end{cases}.$$

Now, in order to find the composition of the two transformation, we find

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

Problem 10

Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution 10

This kind of decomposition is achieved when D has the eigenvalues of A as the elements on its main diagonal and when P is the matrix formed by the corresponding eigenvectors as columns. We begin by finding the characteristic polynomial of A:

$$c_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & -1 & 3-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda)(3-\lambda).$$

We solve for the eigenvalues of A by finding the roots of this polynomial:

$$c_A(\lambda) = 0 \Rightarrow (2-\lambda)(2-\lambda)(3-\lambda) = 0$$

giving us two eigenvalues $\lambda_{1,2} = 2$ (of multiplicity 2) and $\lambda_3 = 3$ (of multiplicity 1). Now we can find the eigenvector corresponding to λ_1 by solving the following:

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

and the eigenvector corresponding to λ_3 by solving the following:

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow X_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Now, we're ready to state the matrices D and P:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Indeed, D is a diagonal and P is invertible ($|P| = -1 \neq 0$).

Problem 11

Let $A = \begin{bmatrix} -1 & 1 \\ -3 & 5/2 \end{bmatrix}$. Calculate A^{1000}.

Solution 11

This kind of problem can be done via the fact that $A = PDP^{-1} \Rightarrow A^{1000} = PD^{1000}P^{-1}$, where D has the eigenvalues of A as the elements on its main diagonal and when P is the matrix formed by the corresponding eigenvectors as columns. We begin by finding the characteristic polynomial of A:

$$c_A(\lambda) = |A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 \\ -3 & 5/2-\lambda \end{vmatrix} = (-1-\lambda)(5/2-\lambda)+3 = \lambda^2 - 3/2\lambda + 1/2.$$

We solve for the eigenvalues of A by finding the roots of this polynomial:

$$c_A(\lambda) = 0 \Rightarrow \lambda^2 - 3/2\lambda + 1/2 = 0 \Rightarrow (\lambda - 1)(\lambda - 1/2) = 0$$

giving us two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1/2$. Now we can find the eigenvector corresponding to λ_1 by solving the following:

$$\left(\begin{array}{cc|c} -2 & 1 & 0 \\ -3 & 3/2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

and the eigenvector corresponding to λ_2 by solving the following:

$$\left(\begin{array}{cc|c} -3/2 & 1 & 0 \\ -3 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow X_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Now, we're ready to state the matrices D and P:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

Indeed, D is a diagonal and P is invertible ($|P| = -1 \neq 0$). We see that $P^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ and $D^{1000} = \begin{bmatrix} 1 & 0 \\ 0 & (1/2)^{1000} \end{bmatrix}$. Therefore,

$$A^{1000} = PD^{1000}P^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1/2)^{1000} \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}.$$

Calculating this explicitely, we reach

$$A^{1000} = \begin{bmatrix} 1 & (1/2)^{999} \\ 2 & 3(1/2)^{1000} \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3 + (1/2)^{998} & 2 - (1/2)^{999} \\ -6 + 3(1/2)^{999} & 4 - 3(1/2)^{1000} \end{bmatrix}.$$