Solutions to test on March 26, 2012. MATH1310

(1) For each integer $n \geq 1$ a number A_n between 1 and 20,000 is drawn at random from a hat (and returned to the hat for the next draw). What is $\lim_{n\to\infty} A_n/n$. **3 marks**

Solution 1. Since $0 \leq A_n/n \leq 20,000/n$ it follows that $\lim_{n\to\infty} 20,000/n = 20,000 \lim_{n\to\infty} 1/n = 20,000 \times 0 = 0$.

(2) For each of the following series determine whether or not it converges and support your answer by referring to a theorem about convergence:

(a) $\sum_{n=1}^{\infty}(n+1)^{-1/2}$ **3 marks**

Solution 2. Since $\sum_{n=1}^{\infty} n^{-1/2} = \sum_{n=1}^{\infty}(1/n)^{1/2}$ is a p-series with $p < 1$ it follows that $\sum_{n=1}^{\infty} n^{-1/2}$ diverges. Using the limit comparison test and the fact that

$$\lim_{n\to\infty}\frac{(n+1)^{-1/2}}{n^{-1/2}} = \lim_{n\to\infty}\frac{n^{1/2}}{(n+1)^{1/2}} = \sqrt{\lim_{n\to\infty}\frac{n}{n+1}} = \sqrt{\lim_{n\to\infty}\frac{1}{1+1/n}} = 1$$

it follows that $\sum_{n=1}^{\infty}(n+1)^{-1/2}$ diverges.

(b) $\sum_{n=1}^{\infty}\frac{(-1)^n \ln(n)}{n}$ **3 marks**

Solution 3. Since $\ln(n)/n \geq \ln(n+1)/(n+1) \geq 0$, this is an alternating series and so it converges. (To see that $\ln(n)/n \geq \ln(n+1)/(n+1)$ it suffices to show that the derivative of $\ln(x)/x$ is negative for $x \geq 3$. But

$$\frac{d}{dx}\ln(x)/x = \frac{1-\ln(x)}{x^2}$$

and so the derivative is negative if $x > e$.)

(c) $\sum_{n=1}^{\infty}\frac{5n^2-n+7}{1+3n^2+4n^4}$ **3 marks**

Solution 4. Since $\sum_{n=1}^{\infty}(1/n)^2$ is a p-series with $p > 1$ it follows that $\sum_{n=1}^{\infty} n^2$ converges. Using the limit comparison test and the fact that

$$\lim_{n\to\infty}\frac{\frac{5n^2-n+7}{1+3n^2+4n^4}}{1/n^2} = \lim_{n\to\infty}\frac{5n^2-n+7}{1+3n^2+4n^4}n^2 = \lim_{n\to\infty}\frac{5n^4-n^3+7n^2}{1+3n^2+4n^4} = \lim_{n\to\infty}\frac{5-1/n+7/n^2}{1/n^4+3/n^2+4} = \frac{5}{4}$$

it follows that $\sum_{n=1}^{\infty}\frac{5n^2-n+7}{1+3n^2+4n^4}$ converges.

(d) $\sum_{n=1}^{\infty} e^{1/n}$ **3 marks**

Solution 5. Since

$$\lim_{n\to\infty} e^{1/n} = e^{\lim_{n\to\infty} 1/n} = e^0 = 1 \neq 0$$

it follows that $\sum_{n=1}^{\infty} e^{1/n}$ diverges.

(3) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty} a_n = L$ and $0 < L < 1$.

(a) What can be said about the divergence or convergence of $\sum_{n=1}^{\infty} a_n$? **3 marks**

Solution 6. Since $\lim_{n\to\infty} a_n \neq 0$ it follows that $\sum_{n=1}^{\infty} a_n$ diverges.

(b) What can be said about the divergence or convergence of $\sum_{n=1}^{\infty} a_n^n$? **3 marks**

Solution 7. Since

$$\lim_{n\to\infty} \sqrt[n]{|a_n^n|} = \lim_{n\to\infty} |a_n| = |L| = L < 1$$

it follows by the root test that $\sum_{n=1}^{\infty} a_n^n$ converges.

(c) If $\lim_{n\to\infty} \ln(a_n) = L$ what can be said about $\lim_{n\to\infty} a_n$? **3 marks**

Solution 8. Since ln is continuous and one-to-one and e^x is the inverse of $\ln(x)$ it follows that $\lim_{n\to\infty} a_n = e^L$.

(4) Let the function F be defined by

$$F(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1)}$$

when the given series converges. For what values of x is F defined?

Solution 9. Using the ration test,

$$\lim_{n\to\infty} \frac{x^{2(n+1)}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1) \cdot (2(n+1)+1)} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1)}{x^{2n}} =$$

$$\lim_{n\to\infty} \frac{x^{2n+2}}{2n+3} \frac{1}{x^{2n}} = \lim_{n\to\infty} \frac{x^2}{2n+3} = 0$$

and so F is defined for all values of x.

Total marks: 34