# Class Test — February 27, 2012.
# MATH1310

*Attempt all questions. No aides are allowed. The time allowed for completion is 45 minutes.*

(1) Use logarithmic differentiation to evaluate the following derivative:

$$\frac{d}{dx}\sqrt{\frac{\sin^3(x)\cos^4(x)}{e^x(x^2-3)}}$$

**4 marks**

**Solution 1.** To use logarithmic differentiation one must take the exponential of ln of the given function. This yields that

$$\frac{d}{dx}\sqrt{\frac{\sin^3(x)\cos^4(x)}{e^x(x^2-3)}} = \frac{d}{dx}e^{\ln\left(\sqrt{\frac{\sin^3(x)\cos^4(x)}{e^x(x^2-3)}}\right)} = e^{\ln\left(\sqrt{\frac{\sin^3(x)\cos^4(x)}{e^x(x^2-3)}}\right)}\frac{d}{dx}\ln\left(\sqrt{\frac{\sin^3(x)\cos^4(x)}{e^x(x^2-3)}}\right) =$$

$$\sqrt{\frac{\sin^3(x)\cos^4(x)}{e^x(x^2-3)}}\frac{1}{2}\frac{d}{dx}\left(\ln(\sin^3(x))+\ln(\cos^4(x))-\ln(e^x)-\ln(x^2-3)\right) =$$

$$\sqrt{\frac{\sin^3(x)\cos^4(x)}{e^x(x^2-3)}}\frac{1}{2}\frac{d}{dx}\left(3\ln(\sin(x))+4\ln(\cos(x))-x-\ln(x^2-3)\right) =$$

$$\sqrt{\frac{\sin^3(x)\cos^4(x)}{e^x(x^2-3)}}\frac{1}{2}\left(\frac{3\cos(x)}{\sin(x)}-\frac{4\sin(x)}{\cos(x)}-1-\frac{2x}{x^2-3}\right) =$$

$$\frac{1}{2}\sqrt{\frac{\sin^3(x)\cos^4(x)}{e^x(x^2-3)}}\left(3\tan(x)-4\cot(x)-1-\frac{2x}{x^2-3}\right)$$

(2) Evaluate the following indefinite integrals:

(a)

$$\int \cos^7(\theta)\sin^4(\theta)d\theta$$

**4 marks**

**Solution 2.** Letting $u=\sin(\theta)$ it follows that $du=\cos(\theta)d\theta$ and so

$$\int \cos^7(\theta)\sin^4(\theta)d\theta = \int \cos^6(\theta)\sin^4(\theta)\cos(\theta)d\theta = \int (1-\sin^2(\theta))^3\sin^4(\theta)\cos(\theta)d\theta =$$

$$\int (1-u^2)^3u^4du = \int (1-3u^2+3u^4-u^6)u^4du = \int u^4-3u^6+3u^8-u^{10}du =$$

$$\frac{\sin^5(\theta)}{5}-\frac{3\sin^7(\theta)}{7}+\frac{3\sin^9(\theta)}{9}-\frac{\sin^{11}(\theta)}{11}+C$$

(b)

$$\int \frac{x+2}{x^3-x^2}dx$$

**4 marks**

**Solution 3.** Begin by finding that $A$, $B$ and $C$ such that

$$\frac{x+2}{x^3-x^2} = \frac{x+2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1)+B(x-1)+Cx^2}{x^2(x-1)}$$

yielding that

$$x+2 = (A+C)x^2 - (A-B)x - B$$

and hence that

$$A+C=0$$
$$A-B=-1$$
$$-B=2$$

It follows that $B=-2$, $A=-3$ and $C=3$. Therefore

$$\int \frac{x+2}{x^3-x^2}dx = \int \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}dx = -3\int \frac{dx}{x} - 2\int \frac{2dx}{x^2} + 3\int \frac{dx}{x-1} = -3\ln|x| + 2/x + 3\ln|x-1| + K =$$
$$3\ln\left|\frac{x-1}{x}\right| + 2/x + K$$

(c) Using the fact that

$$\frac{d}{dx}(x\ln(x)-x) = \ln(x)$$

evaluate

$$\int \ln^2(x)dx$$

**4 marks**

**Solution 4.** Using integration by parts with $u=\ln(x)$ and $dv=\ln(x)dx$ it follows that $du = dx/x$ and that $v = x\ln(x)-x$. Hence

$$\int \ln^2(x)dx = uv - \int vdu = \ln(x)(x\ln(x)-x) - \int \frac{x\ln(x)-x}{x}dx = \ln(x)(x\ln(x)-x) - \int(\ln(x)-1)dx$$
$$= \ln(x)(x\ln(x)-x) - \int \ln(x)dx + x = \ln(x)(x\ln(x)-x) - (x\ln(x)-x) + x = x(\ln(x)-1)^2 + x$$

(3) Evaluate the following definite integral:

$$\int_0^1 \frac{x}{(1-x^2)^{3/2}}dx$$

**6 marks**

**Solution 5.** Since $x/(1-x^2)^{3/2}$ is not defined at $x=1$ the integral is improper and so

$$\int_0^1 \frac{x}{(1-x^2)^{3/2}}dx = \lim_{A\to 1}\int_0^A \frac{x}{(1-x^2)^{3/2}}dx$$

To evaluate the indefinite integral let $u = 1-x^2$ so that $du = -2xdx$. Hence

$$\int \frac{x}{(1-x^2)^{3/2}}dx = \frac{-1}{2}\int \frac{-2x}{u^{3/2}}dx = \frac{-1}{2}\int u^{-3/2}du = \frac{1}{\sqrt{u}} + C = \frac{1}{\sqrt{1-x^2}} + C$$

and so

$$\int_0^1 \frac{x}{(1-x^2)^{3/2}}dx = \lim_{A\to 1}\int_0^A \frac{x}{(1-x^2)^{3/2}}dx = \lim_{A\to 1}\frac{1}{\sqrt{1-x^2}}$$

Since the limit diverges, so does the improper integral.

(4) Define $a^x$ being sure to define any undefined notions used in your definition other than the Riemann integral. **3 marks**

**Solution 6.** See the solutions to Test 1.

**Total marks: 25**