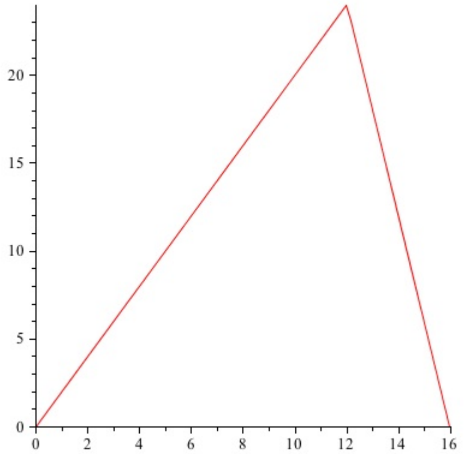
# Class test solutions — January 30, 2012. MATH1310

(1) A car starts from a rest position and accelerates at 2 m/sec$^2$ until it reaches a speed of 24 m/sec. The brakes are then applied and the car slows down at -6 m/sec$^2$ until it stops.

(a) Sketch the velocity function of the car. **4 marks**

**Solution 1.** Since the acceleration is 2 m/sec$^2$ and this is the derivative of the velocity function, the slope of the velocity function is 2 initially. At $t = 0$ the velocity is 0. After the velocity reaches 24m/sec its derivative changes to -6 until it reaches 0. Hence the velocity has the following graph:




(b) How far has the car travelled from stop to finish? **4 marks**

**Solution 2.** The distance travelled is the integral of the velocity function. In this case this is

$$\int_{t=0}^{t=12} 2t dt + \int_{t=12}^{t=16} (96 - 6t) dt$$

and integration yields that this is

$$t^2\Big|_{t=0}^{t=12} - 3t^2\Big|_{t=12}^{t=16} = 12^2 - 3 \times 16^2 + 3 \times 12^2$$

or, alternately, one can interpret the integral as the area of the triangle in the graph. This is $16 \times 24/2 = 16 \times 12 = 192$.

(2) A 3-dimensional solid has cross sections perpendicular to the $x$-axis that are triangles with base $\sqrt{x^3+1}$ and height $2x^2$ for all $x$ such that $0 \leq x \leq 2$. Find the volume of the solid. **6 marks**

**Solution 3.** The volume is the integral of the areas of the cross sections. These areas are given by one half of the base times the height of the triangle — in other words,

$$\sqrt{x^3+1}2x^2/2 = \sqrt{x^3+1}x^2$$

and so the volume is

$$\int_{x=0}^{x=2} \sqrt{x^3+1}x^2 dx$$

Making the chnage of variable $u = x^3 + 1$ yields that $du = 3x^2 dx$ and so the volume is

$$\frac{1}{3}\int_{x=0}^{x=2} \sqrt{x^3+1}3x^2 dx = \int_{x=0}^{x=2} \sqrt{u} du = \int_{u=1}^{u=9} u^{1/2} du = 2u^{3/2}/3\Big|_{u=1}^{u=9} = 2 \times (\sqrt{9})^3/3 - 2/3$$

and this is equal to $2 \times 27/3 = 18 - 2/3 = 17\frac{1}{3}$.

(3) Evaluate the indefinite integral $\int \tan\theta d\theta$. **3 marks**

**Solution 4.** See the book for why this is $\ln|\sec(\theta)|$.

(4) Let $\mathbf{b}$ be a positive number.

(a) Define $\mathbf{b^x}$. **3 marks**

**Solution 5.** See the text for details as to why $\mathbf{b^x} = \exp(\mathbf{x}\ln(\mathbf{b}))$ where $\exp(x)$ is the inverse of $\ln(x)$ and $\ln(x) = \int_1^x dt/t$.

(b) Define $\log_{\mathbf{b}}(x)$. **3 marks**

**Solution 6.** $\log_{\mathbf{b}}(x)$ is the inverse of $\mathbf{b^x}$.

(c) Evaluate the derivative with respect to $x$ of $\log_{\mathbf{b}}(x)$. **3 marks**

**Solution 7.** To evaluate the derivative with respect to $x$ of $\log_{\mathbf{b}}(x)$ use the chain rule and the fact that $\mathbf{b}^{\log_{\mathbf{b}}(\mathbf{x})} = \mathbf{x}$. Note first that

$$\frac{d}{dx}\mathbf{b^x} = \frac{\mathbf{d}}{\mathbf{dx}}\mathbf{e}^{\ln(\mathbf{b^x})} = \mathbf{e}^{\ln(\mathbf{b^x})}\frac{\mathbf{d}}{\mathbf{dx}}\ln(\mathbf{b^x}) = \mathbf{b^x}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}\ln(\mathbf{b}) = \mathbf{b^x}\ln(\mathbf{b})$$

and hence

$$1 = \frac{d}{dx}x = \frac{d}{dx}\mathbf{b}^{\ln_{\mathbf{b}}(\mathbf{x})} = \mathbf{b}^{\ln_{\mathbf{b}}(\mathbf{x})}\ln(\mathbf{b})\frac{\mathbf{d}}{\mathbf{dx}}\ln_{\mathbf{b}}(\mathbf{x}) = \mathbf{x}\ln(\mathbf{b})\frac{\mathbf{d}}{\mathbf{dx}}\ln_{\mathbf{b}}(\mathbf{x})$$

and solving for $\frac{d}{dx}\ln_{\mathbf{b}}(x)$ yields that

$$\frac{d}{dx}\ln_{\mathbf{b}}(x) = \frac{1}{x\ln(\mathbf{b})}$$

**Total marks: 26**