Department of Mathematics, University of Toronto
MAT224H1S - Linear Algebra II
Winter 2013

Problem Set 1 Solutions

1. In the first class we discussed fields and showed that, in addition to the real numbers, the complex numbers form a field. There are of course many others. One of the more important fields in number theory and algebra is $\mathbb{Z}_p$ where p is prime. This field has only p numbers $0, 1, 2, \ldots, (p-1)$ and in this field one evaluates the ordinary sum and product and then takes the remainder after division by p. For example, consider $\mathbb{Z}_2$ one of the smallest and simplest fields. It has only two elements 0 and 1. $1+1=0$ in $\mathbb{Z}_2$ since $1+1=2$ and after dividing 2 by 2 the remainder is 0. In $\mathbb{Z}_2$ then, all possible sums and products are:

$$0+0=0,\ 0+1=1,\ 1+0=1,\ 1+1=0,$$
$$0\cdot 0=0,\ 0\cdot 1=0,\ 1\cdot 0=0,\ 1\cdot 1=1.$$

Write out all possible sums and products for both $\mathbb{Z}_3$ and $\mathbb{Z}_5$. Record the operations of addition and multiplication in a table (see Section 5.1, #11).

Solution:

Tables for $\mathbb{Z}_3$:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Tables for $\mathbb{Z}_5$:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

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2 (a) Consider the subspace $S = span\{(1,2,0,1),(2,0,1,2)\}$ of $\mathbb{Z}_3^4$. Does the vector $(1,1,1,1)$ belong to S? How about the vector $(1,0,1,1)$?

2 (b) Find a basis for the subspace $S = span\{(1,2,1,2,1),(1,1,2,2,1),(0,1,2,0,2)\}$ of $\mathbb{Z}_3^5$.

(**Note**: $\mathbb{Z}_p^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{Z}_p\}$.)

Solution:

(a) By inspection, $(1,1,1,1) = 2(1,2,0,1)+(2,0,1,2)$, so $(1,1,1,1)$ **does** belong to S. (Note that $2+2 = 1$ in $\mathbb{Z}_3$.) On the other hand, if $(1,0,1,1) = a(1,2,0,1) + b(2,0,1,2)$, then by comparing second coordinates we get that $2a = 0$. But this equation has no solutions in $\mathbb{Z}_3$ (as one can see by plugging in $a = 0,1,2$). Consequently, $(1,0,1,1)$ **does not** belong to S.

(b) Place the three spanning vectors of S into a matrix and row reduce it:

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 \end{bmatrix} \xrightarrow{2\times R_2} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 \end{bmatrix}$$
$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{2\times R_3} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As we've obtained 3 leading 1s, we conclude that $\dim S = 3$. Hence a basis for S is given by $\{(1,2,1,2,1),(1,1,2,2,1),(0,1,2,0,2)\}$ (since it consists of 3 elements and already spans S).

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3 (a) Consider the subspace $S = span\{(3,2,4,1),(1,0,3,2),(2,2,0,4)\}$ of $\mathbb{Z}_5^4$. Find the dimension of S.

3 (b) Find the dimension of $P_n(\mathbb{Z}_3)$ for all $n \geq 1$.

Solution:

(a) We proceed as in **2(b)**:

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 2 & 0 & 4 \end{bmatrix} \xrightarrow{2\times R_1, 3\times R_3} \begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{R_2-R_1, R_3-R_1} \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix} \xrightarrow{3\times R_3} \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$
$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So $\dim S =$ number of leading 1s $= 3$.

(b) **Note:** There is a slight ambiguity here. One can interpret the vector space $P_n(\mathbb{Z}_3)$ in two different ways.

On the one hand, one can think of $P_n(\mathbb{Z}_3)$ as consisting of polynomials $\sum_{i=0}^n a_i x^i = a_0 + a_1 x + \cdots + a_n x^n$ of degree $\leq n$ with coefficients a_i in $\mathbb{Z}_3$. Addition and scaling of polynomials is done coefficient-wise and the zero vector is the zero polynomial $0 + 0x + \cdots + 0x^n$. One *defines* two such polynomials $\sum_{i=0}^n a_i x^i$ and $\sum_{i=0}^n b_i x^i$ to be equal if $a_i = b_i$ for all i. So we are treating polynomials just as "formal expressions" — not as functions in any specific sense. In particular, $\sum_{i=0}^n a_i x^i$ is equal to the zero polynomial if and only if $a_i = 0$ for all i. From these remarks, we see that the set $\{1, x, \ldots, x^n\}$ is a basis for $P_n(\mathbb{Z}_3)$. So in this interpretation

$$\dim P_n(\mathbb{Z}_3) = \text{number of elements in this basis} = n+1.$$

On the other hand, one can think of $P_n(\mathbb{Z}_3)$ as consisting of polynomials *viewed as functions on* $\mathbb{Z}_3$, i.e. not just as "formal polynomial expressions" like in the preceding paragraph. In this case, the fact that $x^3 = x \pmod 3$ (check this for $x = 0, 1, 2$) will play a role. Indeed, notice that now, if $n \geq 2$, $\{1, x, x^2\}$ is a basis for $P_n(\mathbb{Z}_3)$, while if $n = 1$ then $\{1, x\}$ is a basis. Thus in this interpretation

$$\dim P_n(\mathbb{Z}_3) = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n \geq 2. \end{cases}$$

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4. Let $A = \begin{bmatrix} 1 & i & -1+i & -1 \\ 2 & 1+2i & -2+3i & -2 \\ 1+i & i & -2+i & -1-i \end{bmatrix}$. Find a basis for the row space of A and the column space of A.

Solution:

Let's row reduce A:

$$A \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & i & -1+i & -1 \\ 0 & 1 & i & 0 \\ 1+i & i & -2+i & -1-i \end{bmatrix} \xrightarrow{R_3-(1+i)R_1} \begin{bmatrix} 1 & i & -1+i & -1 \\ 0 & 1 & i & 0 \\ 0 & 1 & i & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & i & -1+i & -1 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the row space of A is given by the rows of $\mathrm{REF}(A)$ with leading 1s in them:

$$\{(1, i, -1+i, -1), (0, 1, i, 0)\}.$$

A basis for the column space of A is given by the columns of A that correspond to the columns of $\mathrm{REF}(A)$ with leading 1s in them:

$$\{\begin{bmatrix} 1 \\ 2 \\ 1+i \end{bmatrix}, \begin{bmatrix} i \\ 1+2i \\ i \end{bmatrix}\}.$$

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5. Let $T\colon V \to W$ be a linear transformation. Let U be a subspace of W. Show that its pre-image $T^{-1}(U) = \{v \in V \mid T(v) \in U\}$ is a subspace of V.

Solution:

There are three things we must check:

(i) Is 0_V $T^{-1}(U)$? Note that $T(0_V) = 0_W$ is in U, since U is a subspace of W. So 0_V is indeed in $T^{-1}(U)$.

(ii) Is $T^{-1}(U)$ is closed under addition? Suppose that v_1 and v_2 are in $T^{-1}(U)$, which means that $T(v_1)$ and $T(v_2)$ are in U. We wish to show that $v_1 + v_2$ is in $T^{-1}(U)$. That is, we wish to show that $T(v_1 + v_2) \in U$. But

$$T(v_1 + v_2) = \underbrace{T(v_1)}_{\in U} + \underbrace{T(v_2)}_{\in U},$$

which is in U because U, being a subspace, is closed under addition.

(iii) Is $T^{-1}(U)$ is closed under scalar multiplication? Let c be a scalar and let v be in $T^{-1}(U)$. Then $T(v) \in U$. Hence, because U is a subspace, $cT(v) \in U$. But then $T(cv) = cT(v) \in U$. So $cv \in T^{-1}(U)$.

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6. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation that has the matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

relative to the bases $\alpha = \{(1,-1,1),(0,1,0),(1,0,0)\}$ of $\mathbb{R}^3$ and $\beta = \{(3,2),(2,1)\}$ of $\mathbb{R}^2$. Find $T(x,y,z)$ for any $(x,y,z) \in \mathbb{R}^3$.

Solution #1:

From the matrix, we find that

$$\begin{aligned} T(0,1,0) &= [(3,2)]_\beta = 3(3,2) + 2(2,1) = (13,8), \\ T(1,0,0) &= [(1,1)]_\beta = (3,2) + (2,1) = (5,3). \end{aligned}$$

Also,

$$T(0,0,1) = T(1,-1,1)+T(0,1,0)-T(1,0,0) = [(2,1)]_\beta+(13,8)-(5,3) = (2(3,2)+(2,1))+(8,5) = (16,10).$$

Thus

$$\begin{aligned} T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1) &= (5x,3x) + (13y,8y) + (16z,10z) \\ &= (5x+13y+16z, 3x+8y+10z). \end{aligned}$$

Solution #2:

Let $S_{\alpha,\text{std}}$ and $S_{\text{std},\beta}$ denote the change of basis matrices from the standard basis of $\mathbb{R}^3$ to α and from β to the standard basis for $\mathbb{R}^2$, respectively. Then if $[T]$ denotes the standard matrix of T, we have that

$$[T] = S_{\text{std},\beta} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} S_{\alpha,\text{std}}.$$

So let's determine the change of basis matrices.

Let's do $S_{\alpha,\text{std}}$ first:

$$\begin{aligned} [(1,0,0)]_\alpha &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ [(0,1,0)]_\alpha &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ [(0,1,0)]_\alpha &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Thus

$$S_{\alpha,\text{std}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Next, let's do $S_{\text{std},\beta}$:

$$\begin{aligned} [(3,2)]_{\text{std}} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ [(2,1)]_{\text{std}} &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \end{aligned}$$

Thus

$$S_{\text{std},\beta} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

Consequently,

$$[T] = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 13 & 16 \\ 3 & 8 & 10 \end{bmatrix}.$$

This means that

$$T(x,y,z) = \begin{bmatrix} 5 & 13 & 16 \\ 3 & 8 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x + 13y + 16z \\ 3x + 8y + 10z \end{bmatrix},$$

which is exactly what we got in Solution #1.

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7. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation that has the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

relative to the bases $\{(1,2,0),(1,1,1),(1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,1),(1,-1)\}$ of $\mathbb{R}^2$. Find the matrix of T relative to the bases $\{(2,3,0),(1,1,1),(2,3,1)\}$ of $\mathbb{R}^3$ and $\{(3,-1),(1,-1)\}$ of $\mathbb{R}^2$.

Solution:

We proceed along the same lines as in Solution #2 to Problem 6.

Let S_1 denote the change of basis matrix from $\{(2,3,0),(1,1,1),(2,3,1)\}$ to $\{(1,2,0),(1,1,1),(1,1,0)\}$ and let S_2 denote the change of basis matrix from $\{(1,1),(1,-1)\}$ to $\{(3,-1),(1,-1)\}$. Then our desired matrix will be given by

$$S_2 \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} S_1.$$

Let's determine S_1. We have:

$$\begin{aligned} (2,3,0) &= (1,2,0) + 0(1,1,1) + (1,1,0) \\ (1,1,1) &= 0(1,2,0) + (1,1,1) + 0(1,1,0) \\ (2,3,1) &= (1,2,0) + (1,1,1) + 0(1,1,0). \end{aligned}$$

Thus

$$S_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Next, let's determine S_2:

$$\begin{aligned} (1,1) &= (3,-1) - 2(1,-1) \\ (1,-1) &= 0(3,-1) + (1,-1). \end{aligned}$$

Thus

$$S_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Consequently, the matrix we want is

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 3 \end{bmatrix}$$

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8. Let β be a basis for the n-dimensional vector space V over the field F and let $v_1, v_2, \ldots, v_n$ be vectors in V. Prove that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V if and only if $\{[v_1]_\beta, [v_2]_\beta, \ldots, [v_n]_\beta\}$ is a basis for F^n.

Solution:

Let $T \colon V \to F^n$ be the linear transformation defined by $T(v) = [v]_\beta$. *Recall that T is an isomorphism, i.e., is invertible.* In particular, $\ker T = \{0\}$. We will use this fact below.

Suppose now that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. We wish to show that $\{[v_1]_\beta, [v_2]_\beta, \ldots, [v_n]_\beta\}$ is a basis for F^n. As $\dim F^n = n$ and as the set $\{[v_1]_\beta, [v_2]_\beta, \ldots, [v_n]_\beta\}$ contains n elements, it suffices to show that it is linearly independent. Thus suppose we have

$$a_1[v_1]_\beta + a_2[v_2]_\beta + \cdots + a_n[v_n]_\beta = 0.$$

By the definition of T, this equation is simply

$$a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) = 0.$$

As T is linear, we can rewrite this as

$$T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = 0.$$

This shows that $a_1v_1 + a_2v_2 + \cdots + a_nv_n \in \ker T = \{0\}$. That is,

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

But $v_1, \ldots, v_n$ is a basis for V, so in particular is linearly independent. It follows that $a_1 = \ldots = a_n = 0$, as desired.

Conversely, suppose that $\{[v_1]_\beta, [v_2]_\beta, \ldots, [v_n]_\beta\}$ is a basis for F^n. We wish to show that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. As in the preceding paragraph, it suffices to show that this set is linearly independent. So suppose that

$$a_1v_1 + \cdots + a_nv_n = 0.$$

Now apply T to both sides of the equation to get

$$\begin{aligned} T(a_1v_1 + \cdots + a_nv_n) &= T(0) \\ a_1T(v_1) + \cdots + a_nT(v_n) &= 0 \\ a_1[v_1]_\beta + \cdots + a_n[v_n]_\beta &= 0. \end{aligned}$$

But $[v_1]_\beta, \ldots, [v_n]_\beta$ is a basis for F^n, so $a_1 = \cdots = a_n = 0$ as desired.
