YORK UNIVERSITY

Faculty of Science and Engineering
Department of Mathematics and Statistics

MATH 1505 6.00 C
Test #2
Solutions

1. Let $y = f(x)$ be a function whose derivative $f'(x) = x^3(2x-3)^2(x+1)^5(x-7)$. Classify each critical point of $y = f(x)$ as a local maximum, a local minimum, or neither.

   Answer:

   $f'(x) = 0$, whenever $x = -1,\ x = 0,\ x = \frac{3}{2}$ or $x = 7$. Since $f'(x)$ exists for every $x \in \mathbb{R}$, the critical points of $y = f(x)$ are $x = -1,\ x = 0,\ x = \frac{3}{2}$ and $x = 7$.

   $f'(x)$ is negative on $(-\infty, -1)$ and positive on $(-1, 0)$, so the function has a local minimum at $x = -1$.

   $f'(x)$ is positive on $(-1, 0)$ and negative on $(0, \frac{3}{2})$, so the function has a local maximum at $x = 0$.

   $f'(x)$ is negative on the both intervals $(0, \frac{3}{2})$ and $(\frac{3}{2}, 7)$, so the function has neither local maximum nor local minimum at $x = \frac{3}{2}$.

   Finally, $f'(x)$ is negative on $(\frac{3}{2}, 7)$ and positive on $(7, \infty)$, so the function has a local minimum at $x = 7$.

2. Let $y = f(x) = \dfrac{2x-1}{(x-1)^2}$. The first and second derivatives are

   $f'(x) = -\dfrac{2x}{(x-1)^3}$ and $f''(x) = \dfrac{4x+2}{(x-1)^4}$.

   (a) Determine the domain, range, x and y-intercepts, horizontal and vertical asymptotes of $y = f(x)$.

   Answer:

   $\text{Dom}(f(x)) = \{x \in \mathbb{R} \mid (x-1)^2 \neq 0\} = \{x \in \mathbb{R} \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$.

   $f(x) = 0$ implies that $2x - 1 = 0$, that is $x = \frac{1}{2}$. So, $x = \frac{1}{2}$ is an x-intercept.

   $f(0) = -1$, so, the function has y-intercept at $y = -1$.

   $\lim_{x\to\infty} f(x) = 0^+$ and $\lim_{x\to-\infty} f(x) = 0^-$. So, $y = 0$ is a horizontal asymptote.

   $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} f(x) = \infty$. So, $x = 1$ is a vertical asymptote.

(b) Determine the intervals where $f(x)$ is increasing and where it is decreasing. Find the coordinates of the local extremum point(s) of $f(x)$ (if any).

Answer:
$f'(x)$ does not exist at $x = 1$, however $f(x)$ is not defined there. $f'(x) = 0$ implies $2x = 0$, that is $x = 0$. But $f(0) = -1$. So, $(0, -1)$ is the only critical point.
$f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(1, \infty)$; and $f'(x) > 0$ on the interval $(0, 1)$. Hence, $f(x)$ is decreasing on the intervals $(-\infty, 0)$ and $(1, \infty)$, and increasing on the interval $(0, 1)$. Therefore, $(-1, 0)$ is a local minimum point.

(c) Determine the intervals where the graph of $f(x)$ is concave up and where it is concave down. Find the coordinates of inflection point(s) of $f(x)$ (if any).

Answer:
$f''(x)$ does not exist at $x = 1$, however $f(x)$ is not defined there. $f''(x) = 0$ implies $4x + 2 = 0$, that is $x = -\frac{1}{2}$. But $f(-\frac{1}{2}) = -\frac{8}{9}$. So, $(-\frac{1}{2}, -\frac{8}{9})$ is a possible inflection point.
$f''(x) < 0$ on the interval $(-\infty, -\frac{1}{2})$; and $f''(x) > 0$ on the intervals $(-\frac{1}{2}, 1)$ and $(1, \infty)$. Hence, $f(x)$ is concave down on the interval $(-\infty, -\frac{1}{2})$ and concave up on the intervals $(-\frac{1}{2}, 1)$ and $(1, \infty)$. Therefore, $(-\frac{1}{2}, -\frac{8}{9})$ is an inflection point.

(d) Use all the obtained information to sketch the graph of $y = f(x)$.

Answer: From parts (a) and (b) it follows that the function has no local or global maximum, and the local minimum $(-1, 0)$ is also the global minimum of the function. Therefore, the range of $f(x)$ is $[-1, \infty)$.
The scanned copy of the graph of $y = f(x)$ is given on the separate page.

3. (a) Use the geometric interpretation of a definite integral to evaluate $\int_{-2}^{0} \sqrt{4 - x^2}\, dx$.

Answer:
Write the equation of circumference of the radius of 2 centered at the origin $x^2 + y^2 = 4$. Solving the equation for y, we obtain $y = \pm\sqrt{4 - y^2}$. The area of the corresponding circle equals $\pi 2^2 = 4\pi$. So, $\int_{-2}^{0} \sqrt{4 - x^2}\, dx$ is defined as a signed area of the quarter-circle located on the second quadrant, and consequently equals $\frac{1}{4} 4\pi = \pi$.

(b) Given $\int_{0}^{3} f(x)\, dx = -3$ and $\int_{0}^{5} f(x)\, dx = 2$. Use the properties of a definite integral to find $\int_{5}^{3} f(x)\, dx$.

Answer:

$$\int_5^3 f(x)\,dx = \int_5^0 f(x)\,dx + \int_0^3 f(x)\,dx$$

$$= -\int_0^5 f(x)\,dx + \int_0^3 f(x)\,dx = -2 + (-3) = -5.$$

(c) Find the derivative of $F(x)$ if

$$F(x) = \int_{\frac{\pi}{2}}^{x^2} (t - \sin^2 t)\,dt\,.$$

Hint: Use Leibniz's Rule.

Answer:

$$F'(x) = \frac{d}{dx}[\int_{\frac{\pi}{2}}^{x^2} (t - \sin^2 t)\,dt]$$

$$= (x^2 - \sin^2 x)\frac{d}{dx}(x^2) - (\frac{\pi}{2} - \sin^2\frac{\pi}{2})\frac{d}{dx}(\frac{\pi}{2}) = 2x(x^2 - \sin^2 x).$$

4. (a) $\displaystyle\int \frac{2x-3}{x}\,dx =$ ________________

Answer:

$$\int \frac{2x-3}{x}\,dx = \int (2 - \frac{3}{x})\,dx = \int 2\,dx - \int \frac{dx}{x} = 2x - 3\ln|x| + C.$$

(b) $\displaystyle\int \sqrt{1+\sin x}\cos x\,dx =$ ________________

Answer:

Use Substitution Rule,

$$u = 1 + \sin x \implies du = \cos x\,dx\,,$$

we obtain

$$\int \sqrt{1+\sin x}\cos x\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1+\sin x)^{\frac{3}{2}} + C.$$

(c) $\displaystyle\int \frac{2}{x^2+3x}\,dx =$ ________________

Answer:

$$\frac{2}{x^2+3x} = \frac{2}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}.$$

So,

$$2 = A(x+3) + Bx,$$
$$2 = (A+B)x + 3A.$$

Equating the coefficients of x and constant terms,
$A + B = 0$,
$3A = 2$.
Solving the system simultaneously, we obtain $A = \frac{2}{3},\ B = -\frac{2}{3}$.
Hence,

$$\int \frac{2}{x^2+3x}\,dx = \frac{2}{3}\int (\frac{1}{x} - \frac{1}{x+3})\,dx = \frac{2}{3}(\int \frac{dx}{x} - \int \frac{dx}{x+3})$$
$$= \frac{2}{3}(\ln|x| - \ln|x+3|) + C = \frac{1}{3}\ln(\frac{x}{x+3})^2 + C.$$

5. (a) $\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \tan x\,dx =$ __________________

Answer:

$$\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec x \tan x\,dx = \sec x|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{\cos x}|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2 - \frac{2}{\sqrt{3}} = 2(1 - \frac{1}{\sqrt{3}}).$$

(b) $\int\limits_{0}^{\frac{\pi}{6}} e^x \cos x\,dx =$ __________________

Hint: Use integration by parts.
Answer:

$$\int e^x \cos x\,dx = \int e^x\,d(\sin x) = e^x \sin x - \int \sin x\,d(e^x)$$
$$= e^x \sin x - \int e^x \sin x\,dx = e^x \sin x + \int e^x\,d(\cos x) = e^x \sin x + e^x \cos x - \int \cos x\,d(e^x)$$
$$= e^x(\sin x + \cos x) - \int e^x \cos x\,dx.$$

Solving for $\int e^x \cos x\,dx$, we obtain

$$\int e^x \cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x).$$

Hence,

$$\int\limits_{0}^{\frac{\pi}{6}} e^x \cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x)|_0^{\frac{\pi}{6}} = \frac{1}{4}[(1+\sqrt{3})e^{\frac{\pi}{6}} - 2].$$

(c) Sketch the region bounded by the curves $y = 8 - x^2$ and $y = x^2$, and find its area.

Answer:

Equate the functions,

$$8 - x^2 = x^2 \;\Leftrightarrow\; 2x^2 = 8 \;\Leftrightarrow\; x^2 = 4.$$

So, the graphs of the functions intersect at $x = \pm 2$ and for all $x \in (-2, 2)$, $x^2 \leq 8 - x^2$.

Hence, the area of the region bounded by the curves $y = 8 - x^2$ and $y = x^2$ is equal

$$\int_{-2}^{2} (y^{above} - y^{below})\, dx = \int_{-2}^{2} (8 - x^2 - x^2)\, dx = \int_{-2}^{2} (8 - 2x^2)\, dx = 2\int_{0}^{2} (8 - 2x^2)\, dx$$

[because the integrand is an even function]

$$= 2\int_{0}^{2} 2(4 - x^2)\, dx = 4\int_{0}^{2} (4 - x^2)\, dx = 4(4x - \frac{x^3}{3})|_0^2 = 4[(4 \cdot 2 - \frac{2^3}{3}) - 0] = \frac{64}{3}.$$

The scanned copy of the region is given on the separate page.

6. Two industrial plants, A and B, are located 18 km apart, and each day emit 80 ppm (parts per million) and 720 ppm of particulate matter, respectively. Plant A is surrounded by a restricted area of radius 1 km, in which no housing is allowed, while the restricted area around plant B has a radius of 2 km. The concentration of particle matter arriving at any other point Q from each plant decreases proportional to the reciprocal of the distance between that plant and Q. Where should a house be located on a road joining two plants to minimize the total concentration of particulate matter arriving from both plants?

Hint: Recall that the reciprocal of a number a is $\frac{1}{a}$, whenever $a \neq 0$.

Answer:

Let x be the distance from plant A to the point Q on a road joining two plants. Then concentration of particle matter at the point Q is given by

$$P(x) = \frac{80}{x} + \frac{720}{18 - x} = 80(\frac{1}{x} + \frac{9}{18 - x}).$$

We need to minimize $P(x)$ with respect to x on the interval $[1, 16]$.

$$P'(x) = \frac{d}{dx}[80(\frac{1}{x} + \frac{9}{18 - x})] = 80[-\frac{1}{x^2} + \frac{9}{(18 - x)^2}]$$

$$= 80\frac{8x^2 + 36x - 324}{x^2(18 - x)^2} = 320\frac{2x^2 + 9x - 81}{x^2(18 - x)^2}.$$

$P'(x)$ does not exists at $x = 0$ and $x = 18$, but they are not in the interval $[1, 16]$. OTOH, $P'(x) = 0 \iff 2x^2 + 9x - 81 = 0$, that is if and only if $x = 4.5$ or $x = -9$. Since distance cannot be negative, the only critical point in the allowable interval is $x = 4.5$. Evaluating $P(x)$ at $x = 4.5$ and two endpoints of the interval, $x = 1$ and $x = 16$, we obtain

$$P(4.5) \approx 71.08,$$

$$P(1) \approx 122.4,$$

$$P(16) = 365.$$

Hence, $P(x)$ attains its local and global minimum valuue on the interval $[1, 16]$, at the point $x = 4.5$ and therefore, total pollution is minimized when the house is located 4.5 km from plant A.

The end.