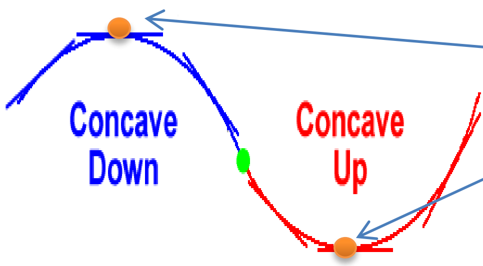
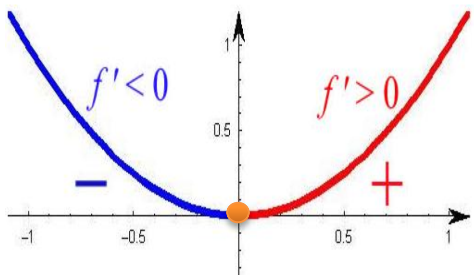
<u>3.3 - Concavity and Second Derivative</u>

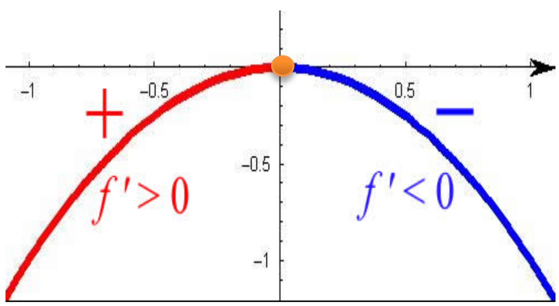
<u>Concavity</u>



A curve can either be concave **up** concave **down.** If we have any maximum or minimum points, around maximum, it is always concave down, and around the local minimum, it is always concave up.



We see that if we move from left to right, on the left side, the derivative is negative and gets less negative as we move along the graph until it reaches 0. After zero, the derivative is now positive and becomes more and more positive as we move along the graph. In this example, f'(x) is increasing.



In this case, we have a downward concave with a maximum point. If we start from the left side, we see that the derivative is positive and as we move along it gets less positive until it hits zero, then starts to become negative and more and more negative as you move along the curve.

From these two graphs we can come up with two important definitions and a theorem to help us understand the concept of concavity:

1. **A function is concave up if f'(x) is increasing on the interval**
2. **A function is concave down if f'(x) is decreasing on the interval.**

Theorem:

If f''(x)>0 for all values the x values on the interval, then f(x) is concave up on the interval. Similarly, if f''(x)<0 for all x values on the interval, then f(x) is concave down on the interval.

Example:

Consider the function **f(x) = x^4 + x^3 + 3x^2 + 1**.

- Find out where the graph is concave up
- Find out where the graph is concave down.

Step 1:

Find the second derivative of the function:

- f'(x) = $4x^3 + 3x^2 – 6x$
- f''(x) = $12x^2 + 6x - 6 = 6(2x-1)(x+1)$
- so x is either 1 or ½

Step 2:

Draw a "number" line with the x values and you will have 3 spaces:

- before the first x-value i.e. -1
- Between the 2 x values i.e. -1 and ½
- After the second x values i.e. ½
    - What you do is basically pick any value within the three spaces, plug them into the second derivative and see whether they are concave up (positive value) or concave down (negative value).
    - So for example, for the first space pick and number between negative 1 and negative infinity to plug in. if you plug in -2 in the second derivative, you get 30, which is a positive number, hence it is concave up. You do this for each space, plugging in values i.e. for second space, any number between negative 1 and ½ and for the 3rd space, any number between half and infinity.

Therefore, it is concave up on the left of the point x = -1 and after the point x = ½

It is concave down between the points x= 1 and x= ½

Places where f(x) switches from one concavity to the other concavity are called **points of inflection**

If there is a point of inflection at x = c, then f''(x) = 0, or f''(x) is undefined. BUT it is possible for f''(c)=0, but x = c is not an inflection point.

After all this has been understood, this brings us to the second test, which is known as the "Second Derivative Test"

Second Derivative Test

Suppose there is a critical point at $x = c$, here are the cases:

1. If $f''(c) > 0$, then there is a local minimum at $x = c$
2. If $f''(c) < 0$, then there is a local maximum at $x = c$
3. If $f''(c) = 0$, then the **second derivative test has failed. (**Hence a different test such as the extreme value test can be used.**)**

That's basically it for the second derivative test.