

# APPLICATIONS OF GAUSSIAN ELIMINATION TO VECTOR SPACES

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Recall that many basic questions about vector spaces can be stated in terms of linear combinations. It is not a surprise that we can obtain specific methods using Gaussian elimination.

## 1. BASIC PROBLEMS ON LINEAR COMBINATIONS

Let  $S = \{v_1, \dots, v_r\}$  be a fixed set of vectors in a vector space  $V$ , where  $V$  is either  $\mathbb{R}^n$ ,  $\mathcal{P}_n$ , or  $\mathcal{M}_{m \times n}$ . We can do the following:

**1.0.1. Determine if a vector  $w$  is in  $\text{Span}\{S\}$ . If it is, we can obtain the coefficients that give us  $w$  as a linear combination of the vectors in  $S$ :**

- (1) Consider the augmented matrix  $(A|w)$ , where the  $i - th$  column of  $A$  is given by the coefficients/entries in  $v_i$ .
- (2) Apply Gaussian elimination to  $(A|w)$ .
- (3) If the obtained system is inconsistent, then  $w$  is not in  $\text{Span}\{S\}$ .
- (4) If the obtained system is consistent, then  $w$  can be written as

$$w = \alpha_1 v_1 + \dots + \alpha_r v_r,$$

where

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_t \end{pmatrix}$$

is a solution of the system.

*Remark 1.* We can apply this process simultaneously for more than one vector  $w$ . See the examples below.

**Example 2.** Determine if the vectors  $\begin{pmatrix} -5 \\ 1 \\ -20 \\ -1 \\ 14 \end{pmatrix}$  and  $\begin{pmatrix} -3 \\ -6 \\ -12 \\ -6 \\ 3 \end{pmatrix}$  are in

$$W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 16 \\ 9 \\ -3 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 20 \\ 9 \\ -6 \end{pmatrix} \right\}$$

Applying Gaussian elimination to the augmented matrix

$$\left( \begin{array}{ccc|cc} 1 & 4 & 5 & -5 & -3 \\ 2 & 9 & 8 & 1 & -6 \\ 4 & 16 & 20 & -20 & -12 \\ 2 & 9 & 9 & -1 & -6 \\ -1 & -3 & -6 & 14 & 3 \end{array} \right)$$

we obtain  $\left( \begin{array}{ccc|cc} 1 & 4 & 5 & -5 & -3 \\ 2 & 9 & 8 & 1 & -6 \\ 4 & 16 & 20 & -20 & -12 \\ 2 & 9 & 9 & -1 & -6 \\ -1 & -3 & -6 & 14 & 3 \end{array} \right) \sim \left( \begin{array}{ccc|cc} 1 & 0 & 0 & -23 & -3 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$

Thus, both vectors are in  $W$ . More precisely,

$$\begin{pmatrix} -5 \\ 1 \\ -20 \\ -1 \\ 14 \end{pmatrix} = -23 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 2 \\ -1 \end{pmatrix} + 7 \begin{pmatrix} 4 \\ 9 \\ 16 \\ 9 \\ -3 \end{pmatrix} - 2 \begin{pmatrix} 5 \\ 8 \\ 20 \\ 9 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} -3 \\ -6 \\ -12 \\ -6 \\ 3 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 2 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 9 \\ 16 \\ 9 \\ -3 \end{pmatrix} + 0 \begin{pmatrix} 5 \\ 8 \\ 20 \\ 9 \\ -6 \end{pmatrix}$$

**Example 3.** Determine if the polynomials

$$P(x) = x, \quad Q(x) = x^3 - x^2 + 3x - 2$$

are in

$$W = \text{Span} \left\{ \begin{array}{l} x^3 + 2x, \\ 2x^3 + x^2 + 3x + 2, \\ -2x^3 + 3x^2 - 7x + 6 \end{array} \right\}$$

Applying Gaussian elimination to the augmented matrix

$$\left( \begin{array}{ccc|cc} 1 & 2 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -1 \\ 2 & 3 & -7 & 1 & 3 \\ 0 & 2 & 6 & 0 & -2 \end{array} \right)$$

we obtain

$$\left( \begin{array}{ccc|cc} 1 & 2 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -1 \\ 2 & 3 & -7 & 1 & 3 \\ 0 & 2 & 6 & 0 & -2 \end{array} \right) \sim \left( \begin{array}{ccc|cc} 1 & 0 & -8 & 0 & 3 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Thus,  $P(x)$  is not in  $W$ , but  $Q(x)$  is in  $W$  and

$$x^3 - x^2 + 3x - 2$$

$$= 3(x^3 + 2x) + (-1)(2x^3 + x^2 + 3x + 2) + (0)(-2x^3 + 3x^2 - 7x + 6).$$

1.0.2. **Given a set of vector  $S$ , obtain a maximal linearly independent set of vectors in  $S$ .**

In particular, we can determine if  $S$  is linearly independent.

As a consequence, from  $S$  we can obtain a basis for  $\text{Span}\{S\}$ :

- (1) Consider the matrix  $A$ , whose  $i - th$  column is given by the coefficients/entries in  $v_i$ .
- (2) Apply Gaussian elimination to  $A$ :

$$A \sim A'.$$

Here we are assuming that  $A'$  is the RREF associated to  $A$ .

- (3) If  $A'$  has at least one column not containing a leading-one, then the set  $S$  is linearly dependent.
- (4) We obtain a maximal linearly independent set from  $S$  by removing from it those vectors which under Gaussian elimination correspond to the columns in  $A'$  not containing a leading-one.

**Example 4.** Determine if  $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 11 \end{pmatrix} \right\}$  is linearly

independent, and find a basis for  $\text{Span}(S)$ .

Applying Gaussian elimination to the matrix  $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ -1 & 3 & 11 \end{pmatrix}$  we

obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ -1 & 3 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

From this we conclude the following:

- $S$  is linearly dependent since the third column in the RREF matrix does not contain a leading one.
- At most we will have two linearly independent vectors out of  $S$ . This maximal linearly independent set from  $S$  will consist of the vectors that after applying Gaussian elimination contain a leading one:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$$

- In particular, we have

$$\begin{pmatrix} 1 \\ -2 \\ 11 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

- The previous comments imply

$$W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 11 \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}.$$

$$\text{Thus, } \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\} \text{ is a basis for } W.$$

**Example 5.** Determine if the set of polynomials

$$S = \left\{ \begin{array}{l} x^3 + 2x, \\ 2x^3 + x^2 + 3x + 2, \\ -2x^3 + 3x^2 - 7x + 6 \end{array} \right\}$$

is linearly independent, and find a basis for  $\text{Span}(S)$ .

Applying Gaussian elimination to the augmented matrix

$$\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 2 & 3 & -7 \\ 0 & 2 & 6 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \\ 2 & 3 & -7 \\ 0 & 2 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- $S$  is linearly dependent since the third column in the RREF matrix does not contain a leading one.
- At most we will have two linearly independent vectors out of  $S$ . This maximal linearly independent set from  $S$  will consist of the vectors that after applying Gaussian elimination contain a leading one:

$$\{x^3 + 2x, \quad 2x^3 + x^2 + 3x + 2\}.$$

- In particular, we have

$$-2x^3 + 3x^2 - 7x + 6 = -8(x^3 + 2x) + 3(2x^3 + x^2 + 3x + 2).$$

- The previous comments imply

$$\begin{aligned} \text{Span} \{x^3 + 2x, \quad 2x^3 + x^2 + 3x + 2, \quad -2x^3 + 3x^2 - 7x + 6\} \\ = \text{Span} \{x^3 + 2x, \quad 2x^3 + x^2 + 3x + 2\}. \end{aligned}$$

Thus,  $\{x^3 + 2x, \quad 2x^3 + x^2 + 3x + 2\}$  is a basis for  $W$ .

1.0.3. If  $S$  is a linearly independent set in  $V$ , and  $\text{Span}\{S\} \subsetneq V$ , we can extend  $S$  to a basis for  $V$ :

- (1) Consider the matrix  $(A|I_n)$ , where the  $i$ -th column of  $A$  is given by the coefficients/entries in  $v_i$ , and the  $i$ -th column of  $I_n$  is given by the coefficients/entries of the standard unit vectors.
- (2) Apply Gaussian elimination to  $(A|I_n)$ . Let  $(A'|B)$  be the matrix in RREF obtained after Gaussian elimination.
- (3) A basis for  $V$  is obtained by adjoining to  $S$  the standard unit vectors in  $I_n$  which after Gaussian elimination contain a leading one of  $(A'|B)$ .

**Example 6.** Extend

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

to a basis for  $\mathbb{R}^4$ .

After Gaussian elimination we obtain

$$\left( \begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \sim \left( \begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

Thus

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for  $\mathbb{R}^4$ .

**Example 7.** Extend the polynomials

$$\{x^2 + 1, \quad -x^2 + 1\}$$

to a basis for  $\mathbb{P}_2$ .

After Gaussian elimination we obtain

$$\left( \begin{array}{cc|ccc} 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left( \begin{array}{cc|ccc} 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & -1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

Thus,

$$\{x^2 + 1, \quad -x^2 + 1, \quad x\}$$

is a basis for  $\mathbb{P}_2$ .

1.0.4. *Remark.* Many different questions about vectors are approached in a similar way. You should read carefully the problem you will be given in order to proceed correctly.

**Example 8.** Let  $A$  be a  $4 \times 4$  matrix whose  $i$ -th column corresponds to a vector  $v_i$  in  $\mathbb{R}^4$ . If

$$A \sim \begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

what can you say about  $S = \{v_1, v_2, v_3, v_4\}$ ?

Answer:

- $v_2 = -2v_1$  and  $v_4 = -3v_1 + 2v_3$ .
- $S$  is linearly dependent and  $\{v_1, v_3\}$  is a linearly independent.
- $\text{Span}\{S\} = \text{Span}\{v_1, v_3\}$ .

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