Linear Algebra - MATH 221 FINAL EXAM

Name:

Instructions: *The Emory Honor Code will be observed.* No calculators are allowed. Show all work to receive full credit, and justify answers. Be as specific and detailed as possible. Please **box** your final answer.

(1) List at least six equivalent statements to the statement that an $n \times n$ matrix A is an invertible matrix.

(2) Define: **eigenvector**.

(3) Given the following matrix:

$$B = \begin{bmatrix} 5 & 7 & 12 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Is the matrix B invertible? Why, or why not?

(b) Is the matrix B diagonalizable? Why, or why not?

(4) Define a **least squares solution** of $A\mathbf{x} = \mathbf{b}$.

(5) Given the following matrix:

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

(a) Determine the characteristic polynomial of A.

(b) Determine the eigenvalues of A.

(c) For each of the eigenvalues determine a basis for the corresponding eigenspace.

(d) (Fill in the blank.) The eigenspace you just found can be described as the _______ of the matrix $A - \lambda I$.

(e) Diagonalize the matrix A. That is determine the factorization of $A = PDP^{-1}$ where D is diagonal. Once you have found P and D, find P^{-1}.

(f) Use this factorization to show how you can easily compute A^5. You do not actually have to do the computation.

(g) If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P then determine the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

(6) The mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$$

is a linear transformation. Find the matrix representation of T relative to the basis $\mathcal{B} = \{1, t, t^2\}$.

(7) Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set H of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{v}$. (Hint: Consider $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$.)

(8) Compute the orthogonal projection of $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ onto the line through $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and the origin.

Write $\mathbf{y}$ as the sum of a vector in Span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

(9) Let W be the subspace spanned by the $\mathbf{u}$'s.

(a) Find the closest point, $\hat{\mathbf{y}}$, to $\mathbf{y}$ in the subspace W.

$$\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u_2} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$$

(b) Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

(c) Find the distance from $\mathbf{y}$ to W.

(10) The given set is a basis for a subspace W^*.

(a) Apply the Gram-Schmidt process to this basis to produce an orthogonal one.

$$\mathbf{x_1} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{x_2} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(b) Now produce an orthonormal basis for W^*.

(c) Find a QR factorization of $A = [\mathbf{x_1}\ \mathbf{x_2}]$.

(11) Describe all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} -6 \\ 2 \\ -10 \end{bmatrix}$$

Compute the least-squares error associated with this solution.